Riemann- Liouville Fractional Multivariate Opial inequalities on spherical shell

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Abstract

Here is introduced the concept of **Riemann- Liouville fractional radial derivative** for a function defined on a spherical shell. Using polar coordinates we are able to derive multivariate Opial type inequalities over a spherical shell of $\Re^N$, $N \geq 2$, by studying the topic in all possibilities. Our results involve one, two, or more functions. We produce also several generalized univariate fractional Opial type inequalities many of these used to achieve our main goals.

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1 Introduction

This work is motivated by articles of Opial [11], Bessack [6], and Anastassiou-Koliha-Pecaric [4], [5], and Anastassiou [2], [3]. We would like to mention

Theorem A (Opial [11], 1960) Let $c > 0$ and $y(x)$ be real, continuously differentiable on $[0, c]$, with $y(0) = y(c) = 0$. Then

$$\int_0^c |y(x)y'(x)|dx \leq \frac{c}{4} \int_0^c \left(y'(x)\right)^2 dx.$$

Equality holds for the function

$$y(x) = x \ on \ [0, c/2]$$

and

$$y(x) = c - x \ on \ [c/2, c].$$

The next result implies **Theorem A** and is very useful to applications.

Theorem B (Bessack [6], 1962) Let $b > 0$. If $y(x)$ is real, continuously differentiable on $[0, b]$, and $y(0) = 0$, then

$$\int_0^b |y(x)y'(x)|dx \leq \frac{b}{2} \int_0^b \left(y'(x)\right)^2 dx.$$

Equality holds only for $y = mx$, where m is a constant.

Opial type inequalities usually find applications in establishing uniqueness of solution of initial value problems for differential equations and their systems, see Willett [16]. In this article we present a series of various Riemann- Liouville fractional multivariate Opial type inequalities over spherical shells. To achieve our goal we use polar coordinates, and we introduce and use the **Riemann- Liouville fractional radial derivative**. We work on the spherical shell, and not on the ball, because a radial derivative can not be defined at zero. So, we reduce the problem to a univariate one.

Consequently we use a large array of univariate Opial type inequalities involving **generalized Riemann- Liouville fractional derivatives**; these are Riemann- Liouville fractional derivatives defined at arbitrary anchor point $a \in \Re$. So we present also a very large set of generalized univariate Riemann- Liouville fractional Opial type inequalities transferred from earlier ones, proved at anchor point zero for the standard Riemann- Liouville fractional derivative. In our results we involve one, two, or several functions. But first we need to develop an extensive background in three parts, then follow the main results in three subsections.

2 Background- I

Here we follow [13], pp. 149-150 and [14], pp. 87-88. Also here $\Re^N$, $N > 1$ denotes the N-tuple of reals $\Re$ and $\mathcal{N}$ denotes the natural numbers. Let us denote by $dx \equiv \lambda_{\Re^N}(dx)$ the Lebesque measure on $\Re^N$, $N > 1$, and $S^{N-1} := \{x \in \Re^N : |x| = 1\}$ the unit sphere on $\Re^N$, where $|\cdot|$ stands for the Euclidean norm in $\Re^N$. Also denote the ball

$$B(0,R) := \{x \in \Re^N : \ |x| < R\} \subseteq \Re^N, \ R > 0,$$

and the spherical shell

$$A := B(0,R_2) - \overline{B(0,R_1)}, \quad 0 < R_1 < R_2.$$

For $x \in \Re^N - \{0\}$ we can write uniquely $x = rw$, where $r = |x| > 0$, and $w = \frac{x}{r} \in S^{N-1}, \ |w| = 1$. Clearly here

$$\Re^N - \{0\} = (0,\infty) \times S^{N-1},$$

also the map

$$\Phi : \Re^N - \{0\} \to S^{N-1} : \ \Phi(x) = \frac{x}{|x|}$$

is continuous.

Also $\overline{A} = [R_1, R_2] \times S^{N-1}$. Let us denote by $dw \equiv \lambda_{S^{N-1}}(w)$ the surface measure on S^{N-1} to be defined as the image under Φ of $N \cdot \lambda_{\Re^N}$ restricted to the Borel class of $B(0,1) - \{0\}$. More precisely the last definition has as follows: let $A \subset S^{N-1}$ be a Borel set, and let

$$\tilde{A} := \{ru : \ 0 < r < 1, \ u \in A\} \subset \Re^N,$$

we define

$$\lambda_{S^{N-1}}(A) = N \cdot \lambda_{\Re^N}(\tilde{A}).$$

Noting $\Phi(rx) = \Phi(x)$, *all* $r > 0$ and $x \in \Re^N - \{0\}$, one can conclude that

$$\int_{B(0,r)-\{0\}} f \circ \Phi(x) dx = r^N \int_{B(0,1)-\{0\}} f \circ \Phi(x) dx$$

and, thus

$$\int_{B(0,r)-\{0\}} f \circ \Phi(x) dx = \frac{r^N}{N} \int_{S^{N-1}} f(w) \lambda_{S^{N-1}}(dw),$$

for all f non-negative and measurable functions on $(S^{N-1}, \mathcal{B}_{S^{N-1}})$, $\mathcal{B}$ stands for the Borel class.

We denote by

$$w_N \equiv \lambda_{S^{N-1}}(S^{N-1}) = \int_{S^{N-1}} dw = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

the **surface area** of S^{N-1} and we get the volume

$$|B(0,r)| = \frac{w_N\ r^N}{N} = \frac{2\pi^{N/2}\ r^N}{N\ \Gamma(N/2)},$$

so that

$$|B(0,1)| = \frac{2\pi^{N/2}}{N\ \Gamma(N/2)}.$$

Clearly here

$$\text{Vol}(A) = |A| = \frac{w_N(R_2^N - R_1^N)}{N} = \frac{2\pi^{N/2}(R_2^N - R_1^N)}{N\ \Gamma(N/2)}.$$

Next, define

$$\psi : (0,\infty) \times S^{N-1} \to \Re^N - \{0\}$$

by $\psi(r,w) := rw$, ψ is one to one and onto function, thus

$$(r,w) \equiv \psi^{-1}(x) = (|x|, \Phi(x))$$

are called the **polar coordinates** of $x \in \Re^N - \{0\}$.

Finally, define the measure R_N on $\big((0,\infty), \mathcal{B}_{(0,\infty)}\big)$ by

$$R_N(\Gamma) = \int_\Gamma r^{N-1} dr,\ \textit{any}\ \Gamma \in \mathcal{B}_{(0,\infty)}.$$

We mention the very important theorem

Theorem 1 (see exercise 6, pp. 149-150 in [13] and Theorem 5.2.2 pp. 87-88 of [14]) We have that $\lambda_{\Re^N} = (R_N \times \lambda_{S^{N-1}}) \circ \psi^{-1}$ on $\mathcal{B}_{\Re^N - \{0\}}$.

In particular, if f is a non-negative Borel measurable function on $\big(\Re^N, \mathcal{B}_{\Re^N}\big)$, then the Lebesque integral

$$\int_{\Re^N} f(x)dx = \int_{(0,\infty)} r^{N-1} \left(\int_{S^{N-1}} f(rw) \lambda_{S^{N-1}}(dw) \right) dr$$

$$= \int_{S^{N-1}} \left(\int_{(0,\infty)} f(rw) r^{N-1} dr \right) \lambda_{S^{N-1}}(dw). \quad (1)$$

Clearly (1) is true for f a Borel integrable function taking values in $\Re$.

Using the facts that:

i) the Lebesque measure of a Lebesque measurable set K equals to the Lebesque measure of a Borel set (i.e there exist M an F_σ and T an G_δ) sets: $M \subset K \subset T$ with $\lambda_{\Re^N}(K) = \lambda_{\Re^N}(M) = \lambda_{\Re^N}(T)$, see [12], p. 62), and

ii) for each g Lebesque measurable function, there exists an f Borel measurable function such that $g = f$ a.e., see [13], p. 145, we get valid that (1) is true for Lebesque integrable functions $f : \Re^N \to \Re$.

We give the important
Proposition 2 Let

$$f : B(0,R) \to \Re, \;\; R > 0,$$

be a Lebesque integrable function. Then

$$\int_{B(0,R)} f(x)dx = \int_{S^{N-1}} \left(\int_0^R f(rw)r^{N-1}dr \right) dw. \tag{2}$$

Proof
Call

$$F(x) := \begin{cases} f(x), & x \in \; B(0,R), \\ \\ 0, & x \in \Re^N - B(0,R). \end{cases}$$

Then apply (1) for F to get easily (2). ■
At last here, we give the main tool for writing this article.
Proposition 3 Let

$$f : A \to \Re,$$

be a Lebesque integrable function, where

$$A := B(0,R_2) - \overline{B(0,R_1)}, \quad 0 < R_1 < R_2.$$

Then

$$\int_A f(x)dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(rw)r^{N-1}dr \right) dw. \tag{3}$$

Proof
Apply (1) for

$$F(x) := \begin{cases} f(x), & x \in A, \\ \\ 0, & x \in \Re^N - A, \end{cases}$$

then (3) is valid. ■

We will also need the following well-known result.

Proposition 4 Let

$$f : [a, b] \to \Re,$$

be a Lebesque integrable function. Then

$$\int_a^b f(z)dz = \int_0^{b-a} f(t + a)dt. \tag{4}$$

So if $f_a(t) := f(a + t)$, the **translation** of f, then

$$\int_a^b f(z)dz = \int_0^{b-a} f_a(t)dt. \tag{5}$$

3 Background- II

Here we define the Riemann- Liouiville fractional derivative we will be using.

Definition 5 (see [4], [5], [7]) Let $\alpha > 0$. For any $f \in L_1(0, x);\ x > 0$, the **Riemann- Liouiville fractional integral** of f of order α is defined by

$$(J_\alpha f)(s) := \frac{1}{\Gamma(\alpha)} \int_0^s (s - t)^{\alpha-1} f(t)dt, \tag{6}$$

all $s \in [0, x]$, and the **Riemann- Liouiville fractional derivative** of f of order α by

$$D^\alpha f(s) := \frac{1}{\Gamma(m - \alpha)} \left(\frac{d}{ds}\right)^m \int_0^s (s - t)^{m-\alpha-1} f(t)dt, \tag{7}$$

where

$$m := [\alpha] + 1,$$

$[\cdot]$ is integral part.

In addition, we set

$$D^0 f := f := J_0 f,$$

$$J_{-\alpha} f \ = D^{\alpha} f, \quad if \quad \alpha > 0,$$

$$D^{-\alpha} f := J_{\alpha} f, \quad if \quad 0 < \alpha \leq 1.$$

If $\alpha \in \mathcal{N}$, then

$$D^{\alpha} f = f^{(\alpha)}$$

the ordinary derivative.

Definition 6 (see [7]) We say that $f \in L_1(0, x)$, has an L_∞ **fractional derivative** $D^{\alpha} f$ in $[0, x]$ if $x > 0$, iff

$$D^{\alpha-k} f \in C([0, x]), \ k = 1, \ldots, m := [\alpha] + 1; \ \alpha > 0,$$

and

$$D^{\alpha-1} f \in AC([0, x]) \ (absolutely \ continuous \ functions),$$

and

$$D^{\alpha} f \in L_\infty(0, x).$$

We mention

Lemma 7 (see [7]) Let

$$\beta > \alpha \geq 0,$$

let

$$f \in L_1(0, x), \ x > 0,$$

have an L_∞ fractional derivative $D^{\beta} f$ in $[0, x]$ and let

$$D^{\beta-k} f(0) = 0 \ for \ \ k = 1, \ldots, [\beta] + 1.$$

Then

$$D^{\alpha} f(s) := \frac{1}{\Gamma(\beta - \alpha)} \int_0^s (s - t)^{\beta-\alpha-1} D^{\beta} f(t) dt, \tag{8}$$

all $s \in [0, x]$.

Clearly here

$$D^{\alpha} f \in AC([0,x]) \ for \ \beta - \alpha \geq 1$$

and

$$D^{\alpha} f \in C([0,x]), \ for \ \beta - \alpha \in (0,1),$$

hence

$$D^{\alpha} f \in L_{\infty}(0,x),$$

and

$$D^{\alpha} f \in L_1(0,x).$$

Next, we define the **generalized Riemann- Liouiville fractional derivative** with arbitrary **anchor point** $a \in \Re$, see [4].

Definition 8 Let $v \geq 0$, define

$$(D_a^v f)(s) := (D^v f_a)(s-a), \quad s \geq a, \tag{9}$$

for $v = 0$ both sides equal to $f(s)$, and for $v = n \in \mathcal{N}$ we easily get that

$$(D_a^n f)(s) = f^{(n)}(s),$$

the ordinary derivative. Clearly here

$$(D_a^v f)(z+a) = (D^v f_a)(z). \tag{10}$$

We will be using $p(s)$ and $D_a^v f(s)$ in $L_{\infty}(a,x), \ x > a, \ a, x \in \Re$. In that case by using (5) we obtain

$$\int_a^w p(y)(D_a^v f)(y)dy = \int_0^{w-a} p(z+a)(D^v f_a)(z)dz, \tag{11}$$

all $a \leq w \leq x, \ a, \ x \in \Re$, which identity we will use a lot in this article.

Our initial intention is to transfer Riemann- Liouiville fractional Opial inequlities, [2], [3], [4], [5] applied to f_a over $[0, w-a]$, for f over $[a,w]$ and use the generalized Riemann- Liouiville fractional derivative. For that we observe that

Lemma 9 $f \in L_1(a,w)$ iff $f_a \in L_1(0, w-a)$, where $w \geq a, \ a, w \in \Re$; $f_a(t) := f(a+t)$.

Proof
We see that

$$\int_a^w |f(z)|dz = \int_0^{w-a} |f_a(t)|dt.$$

■

We need

Lemma 10 Let

$$F(s) := f(s-a),\ a \in \Re$$

be fixed.

Here $f : [0, w-a] \to \Re$, where $w > a$ and $F : [a, w] \to \Re$. Then

(i) $F \in C([a, w])$ iff $f \in C([0, w-a])$,

(ii) $F \in L_\infty(a, w)$ iff $f \in L_\infty(0, w-a)$,

(iii) $F \in AC([a, w])$ iff $f \in AC([0, w-a])$.

Proof
It is based on the fact that the map $g : [a, w] \to [0, w-a]$, such that $g(s) := s - a$ is one to one and onto.

(i) ($\Rightarrow$) Let F continuous, and let $z_n,\ z \in [0, w-a] :\ \ z_n \to z$, i.e. $z_n + a \to z + a$, here $z_n + a,\ z + a \in [a, w]$.

Hence $F(z_n + a) \to F(z + a)$, i.e. $f(z_n) \to f(z)$, proving continuity of f.

($\Leftarrow$) Let f continuous, and let

$$s_n \to s;\ s_n,\ s \in [a, w] \iff s_n - a,\ s - a \in [0, w-a],$$

and

$$s_n - a \to s - a.$$

Hence $f(s_n - a) \to f(s - a)$, i.e. $F(s_n) \to F(s)$. That is F is continuous.

(ii) We see that

$$|F(s)| = |f(s-a)| \leq \|f\|_{\infty,\ [0,w-a]}$$

a.e. in $s \in [a, w]$.

Hence

$$\|F\|_{\infty,\ [a,w]} \leq \|f\|_{\infty,\ [0,w-a]}.$$

Also

$$|f(s-a)| = |F(s)| \leq \|F\|_{\infty,\ [a,w]}$$

a.e. in $s \in [a,w]$.

So that

$$\|f\|_{\infty,\ [0,w-a]} \leq \|F\|_{\infty,\ [a,w]}.$$

I.e.

$$\|F\|_{\infty,\ [a,w]} = \|f\|_{\infty,\ [0,w-a]},$$

proving the claim.

(iii) ($\Rightarrow$) Let F be absolutely continuous, i.e. $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that whenever $(a_1, b_1), \ldots, (a_n, b_n)$ are disjoint open subintervals of $[a,w]$, then

$$\sum_{i=1}^{n} (b_i - a_i) < \delta\ \Rightarrow\ \sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon.$$

Here $(a_i - a, b_i - a) \subset [0, w-a],\ i = 1, \ldots, n$ and also disjoint.

Rewriting the last statement we have

$$\sum_{i=1}^{n} ((b_i - a) - (a_i - a)) < \delta\ \Rightarrow\ \sum_{i=1}^{n} |f(b_i - a) - f(a_i - a)| < \epsilon,$$

that is f is absolutely continuous.

Notice that any open subinterval $(a_i', b_i') \subset [0, w-a]$ has the form $(a_i - a, b_i - a)$, where $(a_i, b_i) \subset [a,w],\ \textit{all}\ \ i = 1, \ldots, n$; by $a_i' = a_i - a,\ b_i' = b_i - a$.

($\Leftarrow$) Assume now f is absolutely continuous, i.e. $\forall\ \epsilon > 0\ \exists\ \delta > 0$: for any $(a_1, b_1), \ldots, (a_n, b_n)$ that are disjoint subintervals of $[0, w-a]$, then

$$\sum_{i=1}^{n} (b_i - a_i) < \delta\ \Rightarrow\ \sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon.$$

The last statement is rewritten as

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 :\ (a_1 + a, b_1 + a), \ldots, (a_n + a, b_n + a) \subset [a,w],$$

then are disjoint open subintervals, then

$$\sum_{i=1}^{n}((b_i+a)-(a_i+a))<\delta \;\Rightarrow\; \sum_{i=1}^{n}|F(b_i+a)-F(a_i+a)|<\epsilon,$$

by

$$f(b_i)=F(b_i+a),\quad f(a_i)=F(a_i+a).$$

Therefore F is absolutely continuous.

Notice again here that any open subinterval $(a_i', b_i') \subset [a,w]$ has the form (a_i+a, b_i+a), where $(a_i,b_i)\subset[0,w-a]$ *all* $i=1,\dots,n$; by $a_i'= a_i+a,\ b_i'=b_i+a$. ■

We need

Lemma 11 Here $a<w,\ a,w\in\Re$. Then

$$p(s)\in L_\infty(a,w)\ iff\ \delta(z):=p(a+z)\in L_\infty(0,w-a).$$

In fact

$$\|\delta\|_{\infty,\ [0,w-a]}=\|p(s)\|_{\infty,\ [a,w]}.$$

Proof

Let $\mathcal{L}^1$ stand for the class of Lebesque measurable sets. Assume $p(s)$ is Lebesque measurable on $[a,w]$. Then for any $c\in\Re$ we have

$$\mathcal{L}^1([a,w])\ni\{x\in[a,w]:\ p(x)\le c\}=\{a+(x-a)\in[a,w]:\ p(a+(x-a))\le c\}=$$

$$a+\{(x-a)\in[0,w-a]:\ p(a+(x-a))\le c\}=a+\{u\in[0,w-a]:\ p(a+u)\le c\}.$$

I.e.

$$\{u\in[0,w-a]:\ \delta(u)\le c\}=\{u\in[0,w-a]:\ p(a+u)\le c\}=$$

$$-a+\{x\in[a,w]:\ p(x)\le c\}\in\mathcal{L}^1([0,w-a]),$$

for all $c\in\Re$. Hence δ is Lebesque measurable on $[0,w-a]$.

Assume now that δ is Lebesque measurable on $[0,w-a]$. Then for any $c\in\Re$ we have

$$\mathcal{L}^1([0,w-a])\ni\{z\in[0,w-a]:\ \delta(z)\le c\}=$$

$$\{z\in[0,w-a]:\ p(a+z)\le c\}=\{(a+z)-a\in[0,w-a]:\ p(a+z)\le c\}=$$

$$-a+\{(a+z)\in[a,w]:\ p(a+z)\leq c\}=-a+\{x\in[a,w]:\ p(x)\leq c\}.$$

I.e.

$$\{x\in[a,w]:\ p(x)\leq c\}=a+\{z\in[0,w-a]:\ \delta(z)\leq c\}\in\mathcal{L}^1\left([a,w]\right),$$

for all $c\in\Re$. Hence $p(s)$ is Lebesque measurable on $[a,w]$. We do have that

$$|\delta(z)|=|p(a+z)|\leq\|p(s)\|_{\infty,\ [a,w]},$$

a.e. $z\in[0,w-a]$. Hence

$$\|\delta\|_{\infty,\ [0,w-a]}\leq\|p(s)\|_{\infty,\ [a,w]}.$$

Also we have

$$|p(a+z)|=|\delta(z)|\leq\|\delta\|_{\infty,\ [0,w-a]},$$

a.e. $z\in[0,w-a]$. Hence

$$\|p(s)\|_{\infty,[a,w]}\leq\|\delta\|_{\infty,[0,w-a]},$$

proving the claim. ∎

We continue with

Definition 12 We say that $f\in L_1(a,w),\ a<w;\ a,w\in\Re$ has an L_∞ **fractional derivative** $D_a^\beta f$ $(\beta>0)$ in $[a,w]$, iff

$$1)\ D_a^{\beta-k}f\in C([a,w]),\ k=1,\ldots,m:=[\beta]+1$$

$$2)\ D_a^{\beta-1}f\in AC([a,w]),$$

and

$$3)\ D_a^\beta f\in L_\infty(a,w).$$

Based on **Lemma 9, 10**, the last **Definition 12** is equivalent step by step to

Definition 13 We say that

$$f_a(s):=f(a+s)\in L_1(0,w-a)$$

has an L_∞ **fractional derivative** $D^\beta f_a$ in $[0, w-a],\ \beta > 0,\ w > a;\ a, w \in \Re$, iff

$$1)\ D^{\beta-k} f_a \in C([0, w-a]),\ k = 1, \ldots, m := [\beta] + 1$$

$$2)\ D^{\beta-1} f_a \in AC([0, w-a]),$$

and

$$3)\ D^\beta f_a \in L_\infty(0, w-a).$$

Definition 14 Here we define for $s \geq a$,

$$D_a^{\beta-m} f(s) = D_a^{\beta-([\beta]+1)} f(s) := D^{\beta-([\beta]+1)} f_a(s-a)$$

$$= J_{([\beta]+1-\beta)} f_a(s-a) =$$

$$\frac{1}{\Gamma([\beta]+1-\beta)} \int_0^{s-a} (s-(a+t))^{[\beta]-\beta} f(a+t)dt, \tag{12}$$

where $f \in L_1(a, w),\ a < w;\ a, w \in \Re$. Notice that

$$0 < [\beta] + 1 - \beta \leq 1.$$

If $f \in L_\infty(a, w)$, then

$$D_a^{\beta-m} f(s) = \frac{1}{\Gamma(m-\beta)} \int_a^s (s-t)^{[\beta]-\beta} f(t)dt.$$

Remark 15 Notice that

$$\left(D_a^{\beta-k} f\right)(a) = D^{\beta-k} f_a(0),\ for\ k = 1, \ldots, [\beta] + 1. \tag{13}$$

Based on **Lemma 7** we get

Lemma 16 Let $\beta > \alpha \geq 0$ and let $f \in L_1(a, w)$ (iff $f_a \in L_1(0, w-a),\ a < w;\ a, w \in \Re$) have an L_∞ fractional derivative $D_a^\beta f$ in $[a, w]$ (iff f_a have an L_∞ fractional derivative $D^\beta f_a$ in $[0, w-a]$), and let

$$\left(D_a^{\beta-k} f\right)(a) = 0, \quad k = 1, \ldots, [\beta] + 1$$

(which is the same as $D^{\beta-k} f_a(0) = 0,\ for\ k = 1, \ldots, [\beta] + 1$).

Then

$$(i)\ D^{\alpha} f_a(s) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^s (s-t)^{\beta-\alpha-1} D^{\beta} f_a(t) dt, \tag{14}$$

all $s \in [0, w-a]$.

$$(ii)\ D_a^{\alpha} f(s) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^s (s-t)^{\beta-\alpha-1} D_a^{\beta} f(t) dt, \tag{15}$$

all $a \leq s \leq w$.

Clearly here

$$D_a^{\alpha} f \text{ is in } AC([a,w]) \text{ for } \beta - \alpha \geq 1$$

and

$$D_a^{\alpha} f \text{ is in } C([a,w]) \text{ for } \beta - \alpha \in (0,1),$$

hence

$$D_a^{\alpha} f \in L_{\infty}(a,w)$$

and

$$D_a^{\alpha} f \in L_1(a,w).$$

Likewise for $D^{\alpha} f_a$ on $[0, w-a]$.

Proof

By **Lemma 7**, and by **Definition 8** and (14), we have

$$\left(D_a^{\alpha} f\right)(s) = \left(D^{\alpha} f_a\right)(s-a) \overset{(14)}{=} \frac{1}{\Gamma(\beta-\alpha)} \int_0^{s-a} ((s-a)-t)^{\beta-\alpha-1} \left(D_a^{\beta} f\right)(t+a) dt =$$

$$\frac{1}{\Gamma(\beta-\alpha)} \int_0^{s-a} (s-(t+a))^{\beta-\alpha-1} \left(D_a^{\beta} f\right)(t+a) dt$$

$$\overset{(4)}{=} \frac{1}{\Gamma(\beta-\alpha)} \int_a^s (s-t)^{\beta-\alpha-1} \left(D_a^{\beta} f\right)(t) dt, \tag{16}$$

proving (15). ■

4 Background- III

We make

Remark 17 Let $f \in L_1(a,w)$, where $a < w;\ a, w \in \Re$. Let $\beta > 0,\ a \leq s \leq w$, by **Definition 8** we have

$$\left(D_a^\beta f\right)(s) = \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{ds}\right)^m \int_0^{s-a} (s-a-t)^{m-\beta-1} f(a+t)dt, \quad (17)$$

where

$$m := [\beta] + 1.$$

If $\beta = 0$, then

$$\left(D_a^\beta f\right)(s) = f(s).$$

Let now

$$F \in L_1(A) = L_1\left([R_1, R_2] \times S^{N-1}\right).$$

For a fixed $w \in S^{N-1}$, define

$$g_w(r) := F(rw) = F(x),$$

where

$$x \in A := B(0, R_2) - \overline{B(0, R_1)},$$

$$0 < R_1 \leq r \leq R_2, \quad r = |x|, \quad w = \frac{x}{r} \in S^{N-1}.$$

By Fubini's theorem

$$g_w \in L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right),$$

for $\lambda_{S^{N-1}}$- almost every $w \in S^{N-1}$.

Call

$$\begin{aligned} K(F) &:= \{w \in S^{N-1} :\ g_w \notin L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right)\} \\ &= \{w \in S^{N-1} :\ F(\cdot\ w) \notin L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right)\}. \quad (18) \end{aligned}$$

That is

$$\lambda_{S^{N-1}}\left(K(F)\right)=0.$$

Of course,

$$\Theta(F):=[R_1,R_2]\times K(F)\subset A$$

and

$$\lambda_{\Re^N}\left(\Theta(F)\right)=0.$$

By (17) then we have

$$\left(D_{R_1}^{\beta}g_w\right)(r)=\frac{1}{\Gamma(m-\beta)}\left(\frac{d}{dr}\right)^m\int_0^{r-R_1}(r-R_1-t)^{m-\beta-1}g_w(R_1+t)dt, \tag{19}$$

where

$$\beta>0,\ m:=[\beta]+1,\ r\in[R_1,R_2].$$

If $\beta=0$, then

$$\left(D_{R_1}^{\beta}g_w\right)(r)=g_w(r).$$

Formula (19) is written for all $w\in S^{N-1}-K(F)$. We set

$$\left(D_{R_1}^{\beta}g_w\right)(r)\equiv 0,\ \forall\ w\in K(F),\forall\ r\in[R_1,R_2],\ any\ \beta>0.$$

The above lead to the following definition.

Definition 18 Let $\beta>0,\ m:=[\beta]+1,\ F\in L_1(A)$, A is the spherical shell. We define

$$\frac{\partial_{R_1}^{\beta}F(x)}{\partial r^{\beta}}:=\begin{cases}\frac{1}{\Gamma(m-\beta)}\left(\frac{d}{dr}\right)^m\int_0^{r-R_1}(r-R_1-t)^{m-\beta-1}F((R_1+t)w)dt, \\ \qquad\qquad\qquad\qquad for\ w\in S^{N-1}-K(F), \\ 0,\quad for\ w\in K(F),\end{cases} \tag{20}$$

where

$$x=rw\in A,\ r\in[R_1,R_2],\ w\in S^{N-1}.$$

If $\beta = 0$, define

$$\frac{\partial_{R_1}^{\beta} F(x)}{\partial r^{\beta}} := F(x).$$

We call

$$\frac{\partial_{R_1}^{\beta} F(x)}{\partial r^{\beta}}$$

the **Riemann- Liouville radial fractional derivative of F of order β.**

We make

Remark 19 If $f \in L_{\infty}(a, w)$, then (17) becomes

$$\left(D_a^{\beta} f\right)(s) = \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{ds}\right)^m \int_a^s (s-t)^{m-\beta-1} f(t) dt, \qquad (21)$$

$\beta > 0,\ m := [\beta] + 1,\ s \in [a, w]$. If F is a Lebesque measurable function from A into $\Re$ and bounded, i.e. there exists

$$M^* > 0 : \ |F(x)| \leq M^*, \ all\ x \in A,$$

then of course

$$F \in L_1(A).$$

Clearly then

$$|g_w(r)| \leq M^*,$$

all $r \in [R_1, R_2]$ and all $w \in S^{N-1}$.

Therefore (19) becomes

$$\left(D_{R_1}^{\beta} g_w\right)(r) = \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{dr}\right)^m \int_{R_1}^r (r-t)^{m-\beta-1} g_w(t) dt, \qquad (22)$$

where

$$m := [\beta] + 1,\ \beta > 0,\ r \in [R_1, R_2],$$

for all $w \in S^{N-1} - K(F)$. In this last case, (20) becomes

$$\frac{\partial^{\beta}_{R_1} F(x)}{\partial r^{\beta}} := \begin{cases} \frac{1}{\Gamma(m-\beta)} \left(\frac{d}{dr}\right)^m \int_{R_1}^r (r-t)^{m-\beta-1} F(tw)dt, \\ \qquad\qquad\qquad\qquad\qquad for\ w \in S^{N-1} - K(F), \\ \\ 0, \quad for\ w \in K(F), \end{cases} \tag{23}$$

where

$$x = rw \in A,\ r \in [R_1, R_2],\ w \in S^{N-1}.$$

We need

Theorem 20 Let $\beta > \alpha > 0$ and $F \in L_1(A)$. Assume that

$$\frac{\partial^{\beta}_{R_1} F(x)}{\partial r^{\beta}} \in L_{\infty}(A).$$

Further assume that $D^{\beta}_{R_1} F(rw)$ takes real values for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D^{\beta}_{R_1} F(rw)| \leq M_1$$

for some $M_1 > 0$.

For each $w \in S^{N-1} - K(F)$, we assume that $F(\cdot\ w)$ have an L_{∞} fractional derivative $D^{\beta}_{R_1} F(\cdot\ w)$ in $[R_1, R_2]$, and that

$$D^{\beta-k}_{R_1} F(R_1 w) = 0,\ k = 1, \ldots, [\beta] + 1.$$

Then

$$\frac{\partial^{\alpha}_{R_1} F(x)}{\partial r^{\alpha}} = \left(D^{\alpha}_{R_1} F\right)(rw) = \frac{1}{\Gamma(\beta-\alpha)} \int_{R_1}^{r} (r-t)^{\beta-\alpha-1} \left(D^{\beta}_{R_1} F\right)(tw)dt, \tag{24}$$

true $\forall\ x \in A$, i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ w \in S^{N-1}$.

Here

$$\left(D^{\alpha}_{R_1} F\right)(\cdot\ w)\ is\ in\ AC([R_1, R_2])\ for\ \beta - \alpha \geq 1$$

and

$$\left(D^{\alpha}_{R_1} F\right)(\cdot\ w)\ is\ in\ C([R_1, R_2])\ for\ \beta - \alpha \in (0, 1),$$

$\forall\ w \in S^{N-1}$.

Furthermore

$$\frac{\partial^{\alpha}_{R_1} F(x)}{\partial r^{\alpha}} \in L_{\infty}(A).$$

In particular, it holds

$$F(x) = F(rw) = \frac{1}{\Gamma(\beta)} \int_{R_1}^{r} (r-t)^{\beta-1} \left(D^{\beta}_{R_1} F \right)(tw) dt, \tag{25}$$

$\forall\ r \in [R_1, R_2]$, $\forall\ w \in S^{N-1} - K(F)$; $x = rw$, and

$$F(\cdot\ w)\ is\ in\ AC([R_1, R_2])\ for\ \beta \geq 1$$

and

$$F(\cdot\ w)\ is\ in\ C([R_1, R_2])\ for\ \beta \in (0,1),$$

$\forall\ w \in S^{N-1} - K(F)$.

Proof

Here we observe that for each $w \in S^{N-1} - K(F)$, we have

$$F(\cdot\ w) \in L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right).$$

By our assumptions and **Lemma 16**, we have

$$\left(D^{\alpha}_{R_1} F\right)(rw) = \frac{1}{\Gamma(\beta-\alpha)} \int_{R_1}^{r} (r-t)^{\beta-\alpha-1} D^{\beta}_{R_1} F(tw) dt, \tag{26}$$

$\forall\ r \in [R_1, R_2]$, $\forall\ w \in S^{N-1} - K(F)$, for $\beta > \alpha \geq 0$. So initially proving (25) by setting $\alpha = 0$ in (26). Here

$$D^{\alpha}_{R_1} F(\cdot\ w)\ is\ in\ AC([R_1, R_2]),\ \forall\ w \in S^{N-1} - K(F),\ \beta - \alpha \geq 1,$$

and

$$D^{\alpha}_{R_1} F(\cdot\ w)\ is\ in\ C([R_1, R_2]),\ for\ \beta - \alpha \in (0,1).$$

Formula (26) for $\alpha > 0$, is true $\forall\ r \in [R_1, R_2]$ and $\forall\ w \in S^{N-1}$, and

$$\left(D^{\alpha}_{R_1} F\right)(\cdot\ w)\ is\ in\ AC([R_1, R_2]),\ \forall\ w \in S^{N-1},\ \beta - \alpha \geq 1$$

and

$$D^{\alpha}_{R_1}F(\cdot\ w)\ is\ in\ C([R_1,R_2]),\ for\ \beta-\alpha\in(0,1).$$

So proving (24). Fixing $r\in[R_1,R_2]$, the function

$$\delta_r(t,w):=(r-t)^{\beta-\alpha-1}D^{\beta}_{R_1}F(tw)$$

is measurable on

$$\left([R_1,r]\times S^{N-1},\overline{\overline{\mathcal{B}}_{[R_1,r]}\times\overline{\mathcal{B}}_{S^{N-1}}}\right).$$

Here $\overline{\overline{\mathcal{B}}_{[R_1,r]}\times\overline{\mathcal{B}}_{S^{N-1}}}$ stands for the complete σ- algebra generated by $\overline{\mathcal{B}}_{[R_1,r]}\times\overline{\mathcal{B}}_{S^{N-1}}$, where $\overline{\mathcal{B}}_X$ stands for the completion of $\mathcal{B}_X$. Then we get that

$$\int_{S^{N-1}}\left(\int_{R_1}^{r}|\delta_r(t,w)|dt\right)dw=\int_{S^{N-1}}\left(\int_{R_1}^{r}(r-t)^{\beta-\alpha-1}|D^{\beta}_{R_1}F(tw)|dt\right)dw\leq$$

$$\left\|\frac{\partial^{\beta}_{R_1}F(x)}{\partial r^{\beta}}\right\|_{\infty,\ ([R_1,r]\times S^{N-1})}\left(\int_{S^{N-1}}\left(\int_{R_1}^{r}(r-t)^{\beta-\alpha-1}dt\right)dw\right)= \tag{27}$$

$$\left\|\frac{\partial^{\beta}_{R_1}F(x)}{\partial r^{\beta}}\right\|_{\infty,\ ([R_1,r]\times S^{N-1})}\left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right)\frac{(r-R_1)^{\beta-\alpha}}{(\beta-\alpha)}\leq$$

$$\left\|\frac{\partial^{\beta}_{R_1}F(x)}{\partial r^{\beta}}\right\|_{\infty,\ A}\left(\frac{2\pi^{N/2}}{\Gamma(N/2)}\right)\frac{(R_2-R_1)^{\beta-\alpha}}{(\beta-\alpha)}<\infty. \tag{28}$$

Hence $\delta_r(t,w)$ is integrable on

$$\left([R_1,r]\times S^{N-1},\overline{\overline{\mathcal{B}}_{[R_1,r]}\times\overline{\mathcal{B}}_{S^{N-1}}}\right).$$

Consequently, by Fubini's theorem and (24), we obtain that

$$\left(D^{\alpha}_{R_1}F\right)(rw),\ \beta>\alpha>0,$$

is integrable in w over $\left(S^{N-1},\ \overline{\mathcal{B}}_{S^{N-1}}\right)$. So we have that

$$\left(D^{\alpha}_{R_1}F\right)(rw),\ \beta>\alpha>0,$$

is continuous in $r\in[R_1,R_2]$ for each $w\in S^{N-1}$, and measurable in $w\in S^{N-1}$ for each $r\in[R_1,R_2]$. So, it is a **Carathéodory function**. Here

$[R_1, R_2]$ is a separable metric space and S^{N-1} is a measurable space, and the function takes values in $\Re^* = \Re \cup \{\pm\infty\}$, which is a metric space. Therefore by **Theorem 20.15**, p.156 of [1],

$$\left(D^{\alpha}_{R_1} F\right)(rw), \ \beta > \alpha > 0$$

is jointly $\left(\mathcal{B}_{[R_1,R_2]} \times \overline{\mathcal{B}}_{S^{N-1}}\right)$- measurable on $[R_1, R_2] \times S^{N-1} = A$, that is Lebesque measurable on A. Indeed then we have that

$$\left| \left(D^{\alpha}_{R_1} F\right)(rw)\right| \leq \frac{1}{\Gamma(\beta-\alpha)} \int_{R_1}^{r} (r-t)^{\beta-\alpha-1} \left| \left(D^{\beta}_{R_1} F\right)(tw) \right| dt$$

$$\leq \frac{\|D^{\beta}_{R_1} F(\cdot\, w)\|_{\infty,\ [R_1,R_2]}}{\Gamma(\beta-\alpha)} \left(\int_{R_1}^{r} (r-t)^{\beta-\alpha-1} dt\right) \leq \frac{M_1}{\Gamma(\beta-\alpha)} \ \frac{(r-R_1)^{\beta-\alpha}}{(\beta-\alpha)} \tag{29}$$

$$\leq \frac{M_1}{\Gamma(\beta-\alpha+1)} \ (R_2 - R_1)^{\beta-\alpha} := \tau < \infty,$$

for all $w \in S^{N-1}$ and all $r \in [R_1, R_2]$. I.e. we proved that

$$\left| \left(D^{\alpha}_{R_1} F\right)(rw)\right| \leq \tau < \infty, \tag{30}$$

for all $w \in S^{N-1}$ and all $r \in [R_1, R_2]$. Hence proving that

$$\frac{\partial^{\alpha}_{R_1} F(x)}{\partial r^{\alpha}} \in L_{\infty}(A).$$

We have finished our proof. ■

We have built the machinery to do Riemann- Liouiville fractional Opial type inequalities on the spherical shell.

Now are ready to present our main results next.

5 Main Results

5.1 Riemann- Liouiville fractional Opial type inequalities involving one function

We mention

Theorem 21 (see [4]) Let

$$\frac{1}{p} + \frac{1}{q} = 1 \ with \ p, q > 1$$

let

$$\gamma \geq 0, \ v > \gamma + 1 - \frac{1}{p},$$

and

$$f \in L_1(0, x)$$

have an L_∞ fractional derivative $D^v f$ in $[0, x], \ x > 0$, such that

$$D^{v-j} f(0) = 0, \ for \ j = 1, \ldots, [v] + 1.$$

Then

$$\int_0^x |D^\gamma f(s)| \ |D^v f(s)| ds \leq \Omega(x) \left(\int_0^x |D^v f(s)|^q ds \right)^{2/q} \tag{31}$$

where

$$\Omega(x) := \frac{x^{(rp+2)/p}}{2^{1/q} \ \Gamma(r+1) \ \left((rp+1)(rp+2)\right)^{1/p}} \tag{32}$$

and

$$r := v - \gamma - 1. \tag{33}$$

We transfer **Theorem 21** to arbitrary anchor point $a \in \Re$. We present **Theorem 22** Let

$$\frac{1}{p} + \frac{1}{q} = 1 \ with \ p, q > 1,$$

let

$$\gamma \geq 0, \ v > \gamma + 1 - \frac{1}{p},$$

and

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x], \ a, \ x \in \Re, \ a < x$, such that

$$D_a^{v-j} f(a) = 0, \ for \ j = 1, \ldots, [v] + 1.$$

Then

$$\int_a^x |D_a^\gamma f(s)|\ |D_a^v f(s)| ds \leq \Omega(x-a) \left(\int_a^x |D_a^v f(s)|^q ds \right)^{2/q} \qquad (34)$$

where Ω as in (32).
Proof
By **Lemma 9**,

$$f_a \in L_1(0, x-a)$$

with L_∞ fractional derivative $D^v f_a$ in $[0, x-a]$, see **Definitions 12, 13**. Furthermore it holds, see (13),

$$D^{v-j} f_a(0) = 0,\ \ for\ j = 1, \ldots, [v] + 1.$$

Therefore by (31) we have

$$\int_0^{x-a} |D^\gamma f_a(s)|\ |D^v f_a(s)| ds \leq \Omega(x-a) \left(\int_0^{x-a} |D^v f_a(s)|^q ds \right)^{2/q}. \qquad (35)$$

Using (10) we have

$$\int_0^{x-a} |\left(D_a^\gamma f\right)(s+a)|\ |\left(D_a^v f\right)(s+a)| ds \leq \Omega(x-a) \left(\int_0^{x-a} |\left(D_a^v f\right)(s+a)|^q ds \right)^{2/q}.$$

By **Lemma 16**, we have that

$$D_a^\gamma f \in AC([a,x])\ for\ v - \gamma \geq 1$$

and

$$D_a^\gamma f \in C([a,x])\ for\ v - \gamma \in (0,1).$$

Clearly then by **Proposition 4** we get

$$\int_0^{x-a} |\left(D_a^\gamma f\right)(s+a)|\ |\left(D_a^v f\right)(s+a)| ds = \int_a^x |D_a^\gamma f(s)|\ |\left(D_a^v f\right)(s)| ds \quad (36)$$

and

$$\int_0^{x-a} |\left(D_a^v f\right)(s+a)|^q ds = \int_a^x |\left(D_a^v f\right)(s)|^q ds,$$

notice here functions under right hand side integrations are integrable. That is proving (34). ■

We mention

Theorem 23 (see [4]) Let

$$v > \gamma \geq 0,$$

and let

$$f \in L_1(0, x)$$

have an L_∞ fractional derivative $D^v f$ in $[0, x],\ x > 0$, such that

$$D^{v-j} f(0) = 0,\ for\ j = 1, \ldots, [v] + 1.$$

Then

$$\int_0^x |D^\gamma f(s)|\ |D^v f(s)| ds \leq \Omega_1(x)\ \text{esssup}_{s \in [0,x]}\ |D^v f(s)|^2, \tag{37}$$

where

$$\Omega_1(x) = \frac{x^{r+2}}{\Gamma(r+3)}, \quad r = v - \gamma - 1. \tag{38}$$

We give the general transfer

Theorem 24 Let

$$v > \gamma \geq 0,$$

and let

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x],\ a,\ x \in \Re,\ a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1.$$

Then

$$\int_a^x |D_a^\gamma f(s)|\ |D_a^v f(s)| ds \leq \Omega_1(x - a)\ \text{esssup}_{s \in [a,x]}\ |D_a^v f(s)|^2, \tag{39}$$

where Ω_1 as in (38).

Proof
By **Lemma 9**,

$$f_a \in L_1(0, x-a)$$

with L_∞ fractional derivative $D^v f_a$ in $[0, x-a]$, see **Definitions 12, 13**. Furthermore it holds, see (13),

$$D^{v-j} f_a(0) = 0, \; for \; j = 1, \ldots, [v]+1.$$

Therefore by (37) we have

$$\int_0^{x-a} |D^\gamma f_a(s)| \, |D^v f_a(s)| ds \leq \Omega_1(x-a) \text{ esssup}_{s\in[0,x-a]} \, |D^v f_a(s)|^2. \quad (40)$$

Using (10) we get

$$\int_0^{x-a} |D_a^\gamma f(s+a)| \, | \left(D_a^v f\right)(s+a)| ds \leq \Omega_1(x-a) \text{esssup}_{s\in[0,x-a]} \, |D_a^v f(s+a)|^2.$$

We have again by **Proposition 4** that

$$\int_0^{x-a} |D_a^\gamma f(s+a)| \, |D_a^v f(s+a)| ds = \int_a^x |D_a^\gamma f(s)| \, |D_a^v f(s)| ds. \quad (41)$$

Also by **Lemma 11** we obtain

$$\text{esssup}_{s\in[0,x-a]} \, |D_a^v f(s+a)|^2 = \text{esssup}_{s\in[a,x]} \, |D_a^v f(s)|^2$$

that is proving the claim. ■

We give the transfer

Theorem 25 Let

$$\frac{1}{p} + \frac{1}{q} = 1 \; with \; 0 < p < 1,$$

let

$$v > \gamma \geq 0,$$

and let

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x],\ a,\ x \in \Re,\ a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1.$$

Additionally assume

$$(1/D_a^v f) \in L_\infty(a,x)$$

and that $D_a^v f$ has the same sign a.e. in (a,x). Then

$$\int_a^x |D_a^\gamma f(s)|\ |D_a^v f(s)| ds \geq \Omega(x-a) \left(\int_a^x |D_a^v f(s)|^q ds \right)^{2/q} \tag{42}$$

where Ω is defined by (32).

Proof

Based on **Theorem 2.3**, of [4], special case of $a = 0$. Similar method of proving as in **Theorem 22**. ■

We give the transfer

Theorem 26 Let

$$\frac{1}{p} + \frac{1}{q} = 1\ with\ p, q > 1$$

let

$$\gamma \geq 0,\ v \geq \gamma + 2 - \frac{1}{p},$$

and

$$f \in L_1(a,x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x],\ a,\ x \in \Re,\ a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1.$$

Then

$$\int_a^x |D_a^\gamma f(s)|\ |D_a^{\gamma+1} f(s)| ds \leq \Omega_2(x-a) \left(\int_a^x |D_a^v f(s)|^q ds \right)^{2/q} \tag{43}$$

where

$$\Omega_2(t) := \frac{t^{2\ (rp+1)/p}}{2\ \left(\Gamma(r+1)\right)^2 (rp+1)^{2/p}}, \quad r = v-\gamma-1, \quad t \geq 0. \tag{44}$$

Proof
Based on **Theorem 2.4** of [4], special case of $a = 0$. Similar method of proving as in **Theorem 22**. ∎

We present the transfer

Theorem 27 Let

$$\gamma \geq 0, \quad v > \gamma + 1,$$

and let

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, $a,\ x \in \Re$, $a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1.$$

Then

$$\int_a^x |D_a^\gamma f(s)|\ |D_a^{\gamma+1} f(s)| ds \leq \Omega_3(x-a)\ \text{esssup}_{t\in(a,x)}\ |D_a^v f(t)|^2, \tag{45}$$

where

$$\Omega_3(t) := \frac{t^{2(v-\gamma)}}{2\ \left(\Gamma(v-\gamma+1)\right)^2}, \quad t \geq 0. \tag{46}$$

Proof
Based on **Theorem 2.5**, of [4], special case of $a = 0$. Similar method of proving as in **Theorem 24**. ∎

We further give

Proposition 28 Inequality (45) is sharp, namely it is attained by

$$f_*(s) := (s-a)^v,\ a \leq s \leq x,\ v > \gamma + 1,\ \gamma \geq 0.$$

Proof
Here we are acting as in **Remark 2.6** of [4]. We use the known formula

$$\int_a^s (s-t)^{u-1} (t-a)^{v-1} dt = \frac{\Gamma(u)\ \Gamma(v)}{\Gamma(u+v)} (s-a)^{u+v-1},\ u, v > 0. \tag{47}$$

Let

$$0 \le j \le [v]+1,\ m := [v]-j+1,\ \alpha := v-[v].$$

We have

$$1-\alpha > 0,\ v+1 > 0,$$

and by (21) we obtain

$$D_a^{v-j} f_*(s) = D_a^{v-j}(s-a)^v = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{ds}\right)^m \int_a^s (s-t)^{(1-\alpha)-1}(t-a)^{(v+1)-1} dt \tag{48}$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(1-\alpha)\ \Gamma(v+1)}{\Gamma(m+j+1)} \left(\frac{d}{ds}\right)^m (s-a)^{m+j} = \frac{\Gamma(v+1)}{j!}(s-a)^j. \tag{49}$$

I.e.

$$D_a^{v-j}(s-a)^v = \frac{\Gamma(v+1)}{j!}(s-a)^j. \tag{50}$$

Hence

$$D_a^{v-j} f_*(a) = 0,\ for\ j = 1, \ldots, [v]+1 \tag{51}$$

and

$$D_a^v f_*(s) = D_a^v (s-a)^v = \Gamma(v+1). \tag{52}$$

Using **Lemma 16**, in particular apply (15), we obtain

$$D_a^\gamma (s-a)^v = \frac{\Gamma(v+1)(s-a)^{v-\gamma}}{\Gamma(v-\gamma+1)}, \quad D_a^{\gamma+1}(s-a)^v = \frac{\Gamma(v+1)}{\Gamma(v-\gamma)}(s-a)^{v-\gamma-1}. \tag{53}$$

Therefore

$$L.H.S(45) = \int_a^x |D_a^\gamma f_*(s)|\ |D_a^{\gamma+1} f_*(s)| ds = \frac{(\Gamma(v+1))^2}{\Gamma(v-\gamma)\ \Gamma(v-\gamma+1)}$$

$$\int_a^x (s-a)^{2(v-\gamma)-1} ds = \frac{1}{2}\left(\frac{\Gamma(v+1)}{\Gamma(v-\gamma+1)}\right)^2 (x-a)^{2(v-\gamma)} = R.H.S(45) \tag{54}$$

That is proving the claim. ■

We present

Theorem 29 Let

$$\frac{1}{p}+\frac{1}{q}=1 \; with \; 0<p<1,$$

let

$$\gamma \geq 0, \; v>\gamma+1,$$

and let

$$f \in L_1(a,x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x], \; a, \; x \in \Re, \; a<x$, such that

$$D_a^{v-j} f(a)=0, \; for \; j=1,\ldots,[v]+1.$$

Also assume

$$(1/D_a^v f) \in L_\infty(a,x)$$

and that $D_a^v f$ has the same sign a.e. in (a,x). Then

$$\int_a^x |D_a^\gamma f(s)| \; |D_a^{\gamma+1} f(s)| ds \geq \Omega_2(x-a) \left(\int_a^x |D_a^v f(s)|^q ds \right)^{2/q}, \tag{55}$$

where Ω_2 is given by (44).

Proof

We transfer here for arbitrary anchor point $a \in \Re$ **Theorem 2.7** of [4]. We apply the earlier established method, see **Theorem 22**. ■

We present

Theorem 30 Let

$$\frac{1}{p}+\frac{1}{q}=1 \; with \; p,q>1,$$

let

$$\gamma \geq 0, \; v>\gamma+1-\frac{1}{p},$$

and let

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, $a,\ x \in \Re,\ a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1;$$

let $m > 0$. Then

$$\int_a^x |D_a^\gamma f(s)|^m ds \leq \Omega_4(x-a) \left(\int_a^x |D_a^v f(s)|^q ds \right)^{m/q} \tag{56}$$

where

$$\Omega_4(t) := \frac{t^{\left(rm+1+\left(\frac{m}{p}\right)\right)}}{(\Gamma(r+1))^m \left(rm+1+\left(\frac{m}{p}\right)\right)(rp+1)^{m/p}},$$

$$r := v - \gamma - 1, \quad t \geq 0. \tag{57}$$

Proof
Based on **Theorem 2.8** of [4]. Its transfer to arbitrary anchor point $a \in \Re$.
∎

We give
Theorem 31 Let

$$v > \gamma \geq 0,$$

and let

$$f \in L_1(a, x)$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, $a,\ x \in \Re,\ a < x$, such that

$$D_a^{v-j} f(a) = 0,\ for\ j = 1, \ldots, [v] + 1;$$

let $m > 0$.
Then

$$\int_a^x |D_a^\gamma f(s)|^m ds \leq \Omega_5(x-a) \operatorname{esssup}_{t \in [a,x]} |D_a^v f(t)|^m, \tag{58}$$

where

$$\Omega_5(t) := \frac{t^{(v-\gamma)m+1}}{(\Gamma(v-\gamma+1))^m \ ((v-\gamma)m+1)},$$

$$t \geq 0. \tag{59}$$

Proof
Based on **Theorem 2.9** of [4], etc. ■

We next give the notation valid for the rest of this subsection 5.1, we follow [5].

Notation 32 Here we call

- l: a positive integer,
- $v, \ r_i$: positive real numbers, $i = 1, \ldots, l,$

$$r = \sum_{i=1}^{l} r_i,$$

- μ_i: real numbers satisfying

$$0 \leq \mu_i < v, \quad i = 1, \ldots, l,$$

- $\alpha_i = v - \mu_i - 1, \quad i = 1, \ldots, l,$
- $\alpha = \max\{(\alpha_i)_- : \quad i = 1, \ldots, l\}$, where $(\alpha_i)_- := (-\alpha_i)_+$,
- $\beta = \max\{(\alpha_i)_+ : \quad i = 1, \ldots, l\}$, where $(\alpha_i)_+ := \max\ (\alpha_i, 0)$,
- w_1, w_2: continuous positive weight functions on $[a, x], \ a, \ x \in \Re, \ a < x$,
- w: continuous nonnegative weight function on $[a, x]$,
- $s_k, s_k' :\ s_k > 0$ and

$$\frac{1}{s_k} + \frac{1}{s_k'} = 1 \ k = 1, 2.$$

We write

$$\overline{\mu} = (\mu_1, \ldots, \mu_l)$$

for a selection of the orders μ_i of fractional derivatives, and

$$\overline{r} = (r_1, \ldots,\ r_l)$$

for a selection of the constants r_i.

We mention

Theorem 33 ([5]) Let

$$f \in L_1(0, x)$$

have an L_∞ fractional derivative $D^v f$ in $[0, x],\ x > 0$, such that

$$D^{v-j} f(0) = 0,\ \ for\ j = 1, \ldots, [v] + 1.$$

Here $a = 0$. For $k = 1, 2$, let $s_k > 1$ and $p > 0$ satisfy

$$\alpha s_2 < 1, \quad p > \frac{s_2}{1 - \alpha s_2} \tag{60}$$

and let

$$\sigma = \frac{1}{s_2} - \frac{1}{p}.$$

Finally, let

$$Q_1 := \left(\int_0^x w_1(\tau)^{s_1'} d\tau \right)^{1/s_1'}$$

$$Q_2 := \left(\int_0^x w_2(\tau)^{-s_2'/p} d\tau \right)^{r/s_2'}. \tag{61}$$

Then

$$\int_0^x w_1(\tau) \prod_{i=1}^{l} |D^{\mu_i} f(\tau)|^{r_i} d\tau \leq$$

$$Q_1 Q_2 C_1\ x^{\rho + \left(\frac{1}{s_1}\right)} \left(\int_0^x w_2(\tau)\ |D^v f(\tau)|^p d\tau \right)^{r/p}, \tag{62}$$

where

$$\rho := \sum_{i=1}^{l} \alpha_i r_i + \sigma r,$$

and

$$C_1 := C_1(v, \overline{\mu}, \overline{r}, p, s_1, s_2) := \frac{\sigma^{r\sigma}}{\prod_{i=1}^{l} \Gamma(v-\mu_i)^{r_i} (\alpha_i+\sigma)^{r_i\sigma} (\rho s_1 + 1)^{1/s_1}}. \tag{63}$$

We transfer last theorem to arbitrary anchor point $a \in \Re$.

Theorem 34 Here all constants and parameters notation is as in **Theorem 33**. Let

$$f \in L_1(a, x), \quad a < x, \; a, x \in \Re,$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, such that

$$D_a^{v-j} f(a) = 0, \; for \; j = 1, \ldots, [v] + 1.$$

Let

$$Q_1(a) := \left(\int_a^x w_1(\tau)^{s_1'} d\tau \right)^{1/s_1'}, \; Q_2(a) := \left(\int_a^x w_2(\tau)^{-s_2'/p} d\tau \right)^{r/s_2'}. \tag{64}$$

Then

$$\int_a^x w_1(\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)|^{r_i} d\tau \leq Q_1(a) Q_2(a) C_1 \, (x-a)^{\rho + \left(\frac{1}{s_1}\right)} \left(\int_a^x w_2(\tau) \, |D_a^v f(\tau)|^p d\tau \right)^{r/p}. \tag{65}$$

Proof

By **Lemma 9**,

$$f_a \in L_1(0, x-a)$$

with an L_∞ fractional derivative $D^v f_a$ in $[0, x-a]$, see **Definitions 12, 13**. Furthermore it holds, see (13),

$$D^{v-j} f_a(0) = 0, \; for \; j = 1, \ldots, [v] + 1.$$

Notice that

$$Q_1(a) := \left(\int_0^{x-a} w_1(\tau + a)^{s_1'} d\tau \right)^{1/s_1'}, \; Q_2(a) := \left(\int_a^{x-a} w_2(\tau + a)^{-s_2'/p} d\tau \right)^{r/s_2'}. \tag{66}$$

Next we apply (62) on $[0, x-a]$ to f_a with respect to

$$w_1(a+\tau),\ w_2(a+\tau),\ \tau \in [0, x-a].$$

We have

$$\int_0^{x-a} w_1(a+\tau) \prod_{i=1}^{l} |D^{\mu_i} f_a(\tau)|^{r_i} d\tau \leq$$

$$Q_1(a) Q_2(a) C_1\ (x-a)^{\rho+\left(\frac{1}{s_1}\right)} \left(\int_0^{x-a} w_2(a+\tau)\ |D^v f_a(\tau)|^p d\tau \right)^{r/p}. \quad (67)$$

Equivalently, via (10), we write

$$\int_0^{x-a} w_1(a+\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(a+\tau)|^{r_i} d\tau \leq$$

$$Q_1(a) Q_2(a) C_1\ (x-a)^{\rho+\left(\frac{1}{s_1}\right)} \left(\int_0^{x-a} w_2(a+\tau)\ |D_a^v f(a+\tau)|^p d\tau \right)^{r/p}. \quad (68)$$

By **Lemma 16**, we have that $D_a^{\mu_i} f \in AC([a, x])$.
Hence, by **Proposition 4**, we get (65). ■
Next, we apply **Theorem 34** to the spherical shell A.
We give

Theorem 35 Here all constants and parameters notation is as in **Theorem 33**. Let $f \in L_1(A)$ with

$$\frac{\partial_{R_1}^v f(x)}{\partial r^v} \in L_\infty(A),\ x \in A;$$

$$A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \Re^N,\ N \geq 2,\ 0 < R_1 < R_2.$$

Further assume that $D_{R_1}^v f(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D_{R_1}^v f(rw)| \leq M_1$$

for some $M_1 > 0$. For each $w \in S^{N-1} - K(F)$, we assume that $f(\cdot\ w)$ has an L_∞ fractional derivative $D_{R_1}^v f(\cdot\ w)$ in $[R_1, R_2]$, and that

$$D_{R_1}^{v-j} f(R_1 w) = 0,\ j = 1, \ldots, [v]+1.$$

We take

$$p=\sum_{i=1}^{l} r_i,$$

and

$$0\leq \mu_1<\mu_2\leq \mu_3\leq \ldots \leq \mu_l<v.$$

If $\mu_1=0$ we set $r_1=1$. Denote

$$Q_1(R_1):=\left(\frac{R_2^{(N-1)s_1'+1}-R_1^{(N-1)s_1'+1}}{(N-1)s_1'+1}\right)^{1/s_1'},$$

$$Q_2(R_1):=\left(\frac{R_2^{(1-N)\frac{s_2'}{p}+1}-R_1^{(1-N)\frac{s_2'}{p}+1}}{(1-N)\frac{s_2'}{p}+1}\right)^{p/s_2'}. \tag{69}$$

Then

$$\int_A \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} dx \leq C^* \left(\int_A \left| \frac{\partial_{R_1}^{v} f(x)}{\partial r^{v}} \right|^{p} dx \right), \tag{70}$$

where

$$C^*:=Q_1(R_1)Q_2(R_1)C_1(R_2-R_1)^{\rho+\left(\frac{1}{s_1}\right)}. \tag{71}$$

Proof
By **Theorem 20** for $\mu_i>0$ we get that

$$\frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \in L_\infty(A).$$

In general here we get that

$$\prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} \in L_1(A).$$

Thus, by **Proposition 3** we have

$$I_1:=\int_A \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} \prod_{i=1}^{l} \left| D_{R_1}^{\mu_i} f(rw) \right|^{r_i} r^{N-1} dr \right) dw =$$

$$\int_{(S^{N-1}-K(f))}\left(\int_{R_1}^{R_2}\prod_{i=1}^{l}|D_{R_1}^{\mu_i}f(rw)|^{r_i}\ r^{N-1}dr\right)dw. \tag{72}$$

Since $\left|\frac{\partial_{R_1}^{v}f(x)}{\partial r^v}\right|^p \in L_1(A)$, we also obtain

$$I_2 := \int_A \left|\frac{\partial_{R_1}^{v}f(x)}{\partial r^v}\right|^p dx = \int_{S^{N-1}}\left(\int_{R_1}^{R_2}|D_{R_1}^{r}f(rw)|^p\ r^{N-1}dr\right)dw =$$

$$\int_{(S^{N-1}-K(f))}\left(\int_{R_1}^{R_2}|D_{R_1}^{r}f(rw)|^p\ r^{N-1}dr\right)dw. \tag{73}$$

Notice here

$$f(\cdot\ w) \in L_1([R_1,R_2]),\ \forall\ w \in S^{N-1}-K(f),$$

and

$$\lambda_{S^{N-1}}(K(f)) = 0.$$

Setting

$$w_1(r) = w_2(r) := r^{N-1},\ r \in [R_1,R_2],$$

we use **Theorem 34**, for every $w \in S^{N-1}-K(f)$.

We get

$$\int_{R_1}^{R_2}\prod_{i=1}^{l}|D_{R_1}^{\mu_i}f(rw)|^{r_i}\ r^{N-1}dr \le$$

$$Q_1(R_1)Q_2(R_1)C_1(R_2-R_1)^{\rho+\left(\frac{1}{s_1}\right)}\left(\int_{R_1}^{R_2}|D_{R_1}^{v}f(rw)|^p\ r^{N-1}dr\right). \tag{74}$$

I.e we found that

$$\int_{R_1}^{R_2}\prod_{i=1}^{l}|D_{R_1}^{\mu_i}f(rw)|^{r_i}\ r^{N-1}dr \le$$

$$C^*\int_{R_1}^{R_2}|D_{R_1}^{v}f(rw)|^p\ r^{N-1}dr,\ \forall\ w \in S^{N-1}-K(f). \tag{75}$$

Therefore

$$\int_{(S^{N-1}-K(f))}\left(\int_{R_1}^{R_2}\prod_{i=1}^{l}|D_{R_1}^{\mu_i}f(rw)|^{r_i}\ r^{N-1}dr\right)dw \leq$$

$$C^*\left(\int_{(S^{N-1}-K(f))}\left(\int_{R_1}^{R_2}|D_{R_1}^{v}f(rw)|^{p}\ r^{N-1}dr\right)dw\right). \tag{76}$$

That is

$$I_1 \leq C^* I_2, \tag{77}$$

so proving (70). ■

We continue with

Theorem 36 Let

$$f \in L_1(a,x), \quad a < x, \ a, x \in \Re$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x]$ such that

$$D_a^{v-j} f(a) = 0, \ for \ j = 1, \ldots, [v] + 1.$$

Then

$$\int_a^x w(\tau)\prod_{i=1}^{l}|D_a^{\mu_i}f(\tau)|^{r_i}d\tau \leq \frac{\|w\|_\infty (x-a)^\rho}{\rho\ \prod_{i=1}^{l}\Gamma(v-\mu_i+1)^{r_i}}\|D_a^v f\|_\infty^r, \tag{78}$$

where

$$\rho := \sum_{i=1}^{l}(v-\mu_i)r_i + 1.$$

Proof

This is a transfer of **Theorem 2.2** of [5] and its proof. By (15) we have

$$|D_a^{\mu_i}f(\tau)| \leq \frac{1}{\Gamma(v-\mu_i)}\int_a^\tau (\tau-t)^{\alpha_i}|D_a^v f(\tau)|dt, \tag{79}$$

implying that

$$|D_a^{\mu_i}f(\tau)| \leq \frac{\|D_a^v f\|_\infty (\tau-a)^{v-\mu_i}}{\Gamma(v-\mu_i+1)}. \tag{80}$$

Hence

$$|D_a^{\mu_i} f(\tau)|^{r_i} \leq \frac{\|D_a^v f\|_\infty^{r_i} \ (\tau - a)^{(v-\mu_i)r_i}}{(\Gamma(v - \mu_i + 1))^{r_i}}, \tag{81}$$

and

$$w(\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)|^{r_i} \leq \frac{\|w\|_\infty \ \|D_a^v f\|_\infty^r \ (\tau - a)^{\sum_{i=1}^{l}(v-\mu_i)r_i}}{\prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}}. \tag{82}$$

Integrating (82) over $[a, x]$ we get

$$\int_a^x w(\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)|^{r_i} d\tau \leq \frac{\|w\|_\infty \ \|D_a^v f\|_\infty^r}{\prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}} \int_a^x (\tau - a)^{\sum_{i=1}^{l}(v-\mu_i)r_i} d\tau =$$

$$\frac{\|w\|_\infty \ \|D_a^v f\|_\infty^r}{\prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}} \ \frac{(x - a)^\rho}{\rho}, \tag{83}$$

proving (78). ■

We apply **Theorem 36** to the spherical shell A case.

Theorem 37 Let

$$f \in L_1(A)$$

with

$$\frac{\partial_{R_1}^v f}{\partial r^v} \in L_\infty(A).$$

Assume $D_{R_1}^v f(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D_{R_1}^v f(rw)| \leq M_1$$

for some $M_1 > 0$.

For each $w \in S^{N-1} - K(f)$, we assume that $f(\cdot\ w)$ has an L_∞ fractional derivative $D_{R_1}^v f(\cdot\ w)$ in $[R_1, R_2]$, and that

$$D_{R_1}^{v-j} f(R_1 w) = 0, \ for \ j = 1, \ldots, [v] + 1.$$

We take

$$0 \leq \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_l < v.$$

If $\mu_1 = 0$ we set $r_1 = 1$.
Then

$$\int_A \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} dx \leq \frac{R_2^{N-1} (R_2 - R_1)^{\rho} M_1^r}{\rho \prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}} \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{84}$$

where

$$\rho := \sum_{i=1}^{l} (v - \mu_i) r_i + 1.$$

Proof
Here

$$\prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} \in L_1(A).$$

Hence as before

$$I_1 := \int_A \prod_{i=1}^{l} \left| \frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}} \right|^{r_i} dx = \int_{(S^{N-1} - K(f))} \left(\int_{R_1}^{R_2} \prod_{i=1}^{l} |D_{R_1}^{\mu_i} f(rw)|^{r_i} \ r^{N-1} dr \right) dw. \tag{85}$$

Here we set

$$w(r) := r^{N-1}, \ r \in [R_1, R_2].$$

For each $w \in S^{N-1} - K(f)$ we apply **Theorem 36**. From (78) we obtain

$$\int_{R_1}^{R_2} \prod_{i=1}^{l} |D_{R_1}^{\mu_i} f(rw)|^{r_i} \ r^{N-1} dr \leq \frac{R_2^{N-1} (R_2 - R_1)^{\rho}}{\rho \prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}} \|D_{R_1}^{v} f(\cdot \, w)\|_{\infty, \ [R_1, R_2]}^{r} \tag{86}$$

$$\leq \frac{R_2^{N-1} (R_2 - R_1)^{\rho} \ M_1^r}{\rho \prod_{i=1}^{l} (\Gamma(v - \mu_i + 1))^{r_i}} := \theta, \ \forall \ w \in S^{N-1} - K(f). \tag{87}$$

Therefore

$$I_1 = \int_{(S^{N-1}-K(f))} \left(\int_{R_1}^{R_2} \prod_{i=1}^{l} |D_{R_1}^{\mu_i} f(rw)|^{r_i} \ r^{N-1} dr \right) dw$$

$$\leq \theta \int_{(S^{N-1}-K(f))} dw = \theta \int_{S^{N-1}} dw \tag{88}$$

$$= \theta \ \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{89}$$

proving (84). ■

We continue with

Theorem 38 Let

$$f \in L_1(a,x), \ a < x, \ a, x \in \Re,$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x]$ such that

$$D_a^{v-j} f(a) = 0, \ for \ j = 1, \ldots, [v]+1.$$

Assume also that $D_a^v f$ has the same sign a.e. in (a,x). For $k = 1, 2$, let

$$0 < s_k < 1,$$

let

$$p < 0$$

and let

$$\sigma := \frac{1}{s_2} - \frac{1}{p},$$

$Q_1(a)$ and $Q_2(a)$ as in (64), C_1 as in (63).

Then

$$\int_a^x w(\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)|^{r_i} d\tau \geq$$

$$Q_1(a) Q_2(a) C_1 \ (x-a)^{\rho + \left(\frac{1}{s_1}\right)} \left(\int_a^x w_2(\tau) |D_a^v f(\tau)|^p d\tau \right)^{r/p}, \tag{90}$$

where

$$\rho := \sum_{i=1}^{l} \alpha_i r_i + \sigma r. \tag{91}$$

Proof
Similar to **Theorem 34**. Transfer of **Theorem 2.3** of [5] to anchor point $a \in \Re$. ■

We give

Theorem 39 Let

$$s_1, s_2, p \in (0,1), \ \ rs_1 \leq 1$$

and

$$\frac{s_2}{1 - \alpha s_2 + s_2} < p < \frac{s_2}{1 + \beta s_2},$$

$$\sigma = \frac{1}{s_2} - \frac{1}{p},$$

ρ as in (91), $Q_1(a)$ and $Q_2(a)$ as in (64), C_1 as in (63).

Let

$$f \in L_1(a, x), \quad a < x, \ \ a, x \in \Re$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, such that

$$D_a^{v-j} f(a) = 0, \ \ for \ j = 1, \ldots, [v] + 1.$$

Assume also that $D_a^v f$ has the same sign a.e. in (a, x). Then

$$\int_a^x w_1(\tau) \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)|^{r_i} d\tau \geq$$

$$Q_1(a) Q_2(a) C_1 \ (x-a)^{\rho + \left(\frac{1}{s_1}\right)} \left(\int_a^x w_2(\tau) |D_a^v f(\tau)|^p d\tau \right)^{r/p}. \tag{92}$$

Proof
Similar transfer of **Theorem 2.4** from [5]. ■

We apply **Theorem 39** on the spherical shell A.

Theorem 40 All parameters and constants are as in **Theorem 39**. We take

$$p=\sum_{i=1}^{l} r_i,$$

and

$$0\le \mu_1<\mu_2\le \mu_3\le \ldots \le \mu_l<v.$$

If $\mu_1=0$ we set $r_1=1$. Let $f\in L_1(A)$ with

$$\frac{\partial_{R_1}^{v} f}{\partial r^{v}}\in L_\infty(A).$$

Assume that $D_{R_1}^{v} f(rw)\in \Re$ for almost all $r\in[R_1,R_2]$, for each $w\in S^{N-1}$, and for these

$$|D_{R_1}^{v} f(rw)|\le M_1$$

for some $M_1>0$.

For each $w\in S^{N-1}-K(F)$, we assume that $f(\cdot\ w)$ has an L_∞ fractional derivative $D_{R_1}^{v} f(\cdot\ w)$ in $[R_1,R_2]$, and that

$$D_{R_1}^{v-j} f(R_1 w)=0,\ j=1,\ldots,[v]+1,$$

also $D_{R_1}^{v}(\cdot\ w)$ has the same sign a.e. in $[R_1,R_2]$.

Then

$$\int_A \prod_{i=1}^{l}\Big|\frac{\partial_{R_1}^{\mu_i} f(x)}{\partial r^{\mu_i}}\Big|^{r_i} dx\ge C^*\left(\int_A \Big|\frac{\partial_{R_1}^{v} f(x)}{\partial r^{v}}\Big|^{p} dx\right), \tag{93}$$

where

$$C^*:=Q_1(R_1)Q_2(R_1)C_1(R_2-R_1)^{\rho+\left(\frac{1}{s_1}\right)}. \tag{94}$$

Proof
Similar to **Theorem 35** by using (92). ■

We continue with

Theorem 41 Let

$$f\in L_1(a,x),\ a<x,\ a,x\in\Re,$$

have an L_∞ fractional derivative $D_a^v f$ in $[a,x]$, such that

$$D_a^{v-j} f(a) = 0, \ for \ j = 1, \ldots, [v] + 1.$$

Let

$$v > \mu_2 \geq \mu_1 + 1 \geq 1.$$

If

$$p, q > 1 : \ \frac{1}{p} + \frac{1}{q} = 1,$$

then

$$\int_a^x |D_a^{\mu_1} f(\tau)| \, |D_a^{\mu_2} f(\tau)| d\tau \leq \ C_2 (x-a)^{2v-\mu_1-\mu_2-1+\left(\frac{2}{q}\right)} \left(\int_a^x |D_a^v f(\tau)|^p d\tau \right)^{2/p}, \tag{95}$$

where

$$C_2 = C_2(v, \mu_1, \mu_2, p)$$

is given by

$$C_2 := \frac{\left(\frac{1}{2}\right)^{(1/p)}}{\Gamma(v-\mu_1)\Gamma(v-\mu_2+1) \left((v-\mu_1)q+1\right)^{1/q} \left((2v-\mu_1-\mu_2-1)q+2\right)^{1/q}} \tag{96}$$

Proof
Transfer to $a \in \Re$ of **Theorem 2.5**, [5]. ■

We give

Theorem 42 Let

$$f \in L_1(a, x), \ a < x, \ a, x \in \Re,$$

have an L_∞ fractional derivative $D_a^v f$ in $[a, x]$, such that

$$D_a^{v-j} f(a) = 0, \ for \ j = 1, \ldots, [v] + 1,$$

also $|D_a^v f|$ is decreasing on $[a, x]$. Let $l \geq 2$. If

$$p, q > 1 : \ \frac{1}{p} + \frac{1}{q} = 1$$

and

$$\left(\sum_{i=1}^{l} \alpha_i\right) p > -1,$$

then

$$\int_a^x \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)| d\tau \leq \ C_3 (x-a)^{(\gamma p + lp + 1)/p} \left(\int_a^x |D_a^v f(t)|^{lq} dt\right)^{1/q}, \quad (97)$$

where

$$\gamma := \sum_{i=1}^{l} \alpha_i$$

and

$$C_3 = C_3(v, \overline{\mu}, p) := \frac{p}{(\gamma p + 1)^{1/p} (\gamma p + p + 1) \prod_{i=1}^{l} \Gamma(v - \mu_i)}. \quad (98)$$

Proof
Transfer of **Theorem 2.6** of [5]. ■

We finish this subsection with

Theorem 43 All as in **Theorem 42** with $p = 1$ and $q = \infty$.

Then

$$\int_a^x \prod_{i=1}^{l} |D_a^{\mu_i} f(\tau)| d\tau \leq C_4 (x-a)^{\gamma + l + 1} \ \|D_a^v f\|_\infty^l, \quad (99)$$

where

$$\gamma := \sum_{i=1}^{l} \alpha_i$$

and

$$C_4 = C_4(v, \overline{\mu}) := \frac{1}{(\gamma + 1)(\gamma + l + 1) \prod_{i=1}^{l} \Gamma(v - \mu_i)}. \quad (100)$$

Proof
Transfer of **Theorem 2.7**, [5]. ■

5.2 Riemann- Liouiville fractional Opial type inequalities involving two functions

We present

Theorem 44 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2,\ \beta - \alpha_i > (1/p), p > 1, i = 1, 2,$$

and let

$$f_1, f_2 \in L_1(a, x),\ a,\ x \in \Re,\ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a, x]$, and let

$$D_a^{\beta-k} f_i(a) = 0,\ for\ k = 1, \ldots, [\beta] + 1;\ i = 1, 2.$$

Consider also

$$p(t) > 0 \ and \ q(t) \geq 0,$$

with all

$$p(t), \frac{1}{p(t)}, q(t) \in L_\infty(a, x).$$

Let

$$\lambda_\beta > 0 \ and \ \lambda_{\alpha_1}, \lambda_{\alpha_2} \geq 0,$$

such that

$$\lambda_\beta < p.$$

Set

$$P_i(s) := \int_0^s (s-t)^{\frac{p(\beta-\alpha_i-1)}{p-1}} (p(t+a))^{-1/(p-1)} \, dt,\ i = 1, 2;\ 0 \leq s \leq x - a, \tag{101}$$

$$A(s) := \frac{q(s+a) \, (P_1(s))^{\lambda_{\alpha_1}\left(\frac{p-1}{p}\right)} \ (P_2(s))^{\lambda_{\alpha_2}\left(\frac{p-1}{p}\right)} \, (p(s+a))^{-\lambda_\beta/p}}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}} \ (\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_2}}}, \tag{102}$$

$$A_0(x-a) := \left(\int_0^{x-a} (A(s))^{p/(p-\lambda_\beta)} \, ds \right)^{(p-\lambda_\beta)/p}, \tag{103}$$

and

$$\delta_1 := \begin{cases} 2^{1-\left((\lambda_{\alpha_1}+\lambda_\beta)/p\right)}, & if \quad \lambda_{\alpha_1}+\lambda_\beta \leq p, \\ \\ 1, \quad if \quad \lambda_{\alpha_1}+\lambda_\beta \geq p. \end{cases} \tag{104}$$

If $\lambda_{\alpha_2} = 0$ we obtain that,

$$\int_a^x q(s) \left[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}} \ |D_a^\beta f_1(s)|^{\lambda_\beta} + | \ D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}} \ |D_a^\beta f_2(s)|^{\lambda_\beta} \right] ds \leq$$

$$\left(A_0(x-a)\Big|_{\lambda_{\alpha_2}=0} \right) \left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta} \right)^{\lambda_\beta/p} \delta_1 \Big[\int_a^x p(s) \left[|D_a^\beta f_1(s)|^p + |D_a^\beta f_2(s)|^p \right] ds \Big]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{105}$$

Proof
Similar proof like of the **Theorems 22 and 34**. Here we transfer **Theorems 4** of [2] to an arbitrary anchor point $a \in \Re$. In fact for $a = 0$ inequality (105) is identical to inequality (8) of **Theorems 4** of [2]. We apply it here for the translates

$$f_{1a} := f_1(\cdot \ + a), \ f_{2a} := f_2(\cdot \ + a)$$

$$p(\cdot \ + a), \ q(\cdot \ + a)$$

and the fractional derivatives

$$D^\beta f_{1a}, \ D^\beta f_{2a}, \ D^{\alpha_i} f_{1a}, \ D^{\alpha_i} f_{2a}, \ i = 1, 2,$$

all over $[0, x-a]$.

We use **Lemma 9**, the equivalent **Definitions 12, 13** and (13). We use (9), (10), (11) too. We get the result by **Proposition 4** applied at the end. ■

We continue with

Theorem 45 All here as in **Theorem 44**. Denote

$$\delta_3 := \begin{cases} 2^{\lambda_{\alpha_2}/\lambda_\beta} - 1, & if \quad \lambda_{\alpha_2} \geq \lambda_\beta, \\ \\ 1, \quad if \quad \lambda_{\alpha_2} \leq \lambda_\beta. \end{cases} \tag{106}$$

If $\lambda_{\alpha_1} = 0$, then it holds

$$\int_a^x q(s) \left[|D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_2}} \ |D_a^\beta f_1(s)|^{\lambda_\beta} + | \ D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} \ |D_a^\beta f_2(s)|^{\lambda_\beta} \right] ds \leq$$

$$\left(A_0(x-a)\Big|_{\lambda_{\alpha_1}=0}\right) 2^{(p-\lambda_\beta)/p}\left(\frac{\lambda_\beta}{\lambda_{\alpha_2}+\lambda_\beta}\right)^{\lambda_\beta/p}\delta_3^{\lambda_\beta/p}$$

$$\left(\int_a^x p(s)\left[|D_a^\beta f_1(s)|^p+|D_a^\beta f_2(s)|^p\right]ds\right)^{\frac{\left(\lambda_{\alpha_2}+\lambda_\beta\right)}{p}}. \tag{107}$$

Proof
Transfer of **Theorem 5** of [2] to $a\in\Re$. Similar proof to **Theorem 44**. ■

The complete case $\lambda_{\alpha_1},\ \lambda_{\alpha_2}\neq 0$ follows.

Theorem 46 All here as in **Theorem 44**. Denote

$$\tilde{\gamma_1}:=\begin{cases}2^{((\lambda_{\alpha_1}+\lambda_{\alpha_2})/\lambda_\beta)}-1, & if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}\geq\lambda_\beta,\\ 1, \quad if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}\leq\lambda_\beta,\end{cases} \tag{108}$$

and

$$\tilde{\gamma_2}:=\begin{cases}1, \quad if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta\geq p,\\ 2^{1-((\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)/p)}, & if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta\leq p.\end{cases} \tag{109}$$

Then, it holds

$$\int_a^x q(s)\Big[|D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_2(s)|^{\lambda_{\alpha_2}}\ |D_a^\beta f_1(s)|^{\lambda_\beta}$$

$$+|\ D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\alpha_1}f_2(s)|^{\lambda_{\alpha_1}}\ |D_a^\beta f_2(s)|^{\lambda_\beta}\Big]ds$$

$$\leq A_0(x-a)\left(\frac{\lambda_\beta}{(\lambda_{\alpha_1}+\lambda_{\alpha_2})(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)}\right)^{\lambda_\beta/p}\cdot[\lambda_{\alpha_1}^{\lambda_\beta/p}\tilde{\gamma_2}+2^{(p-\lambda_\beta)/p}\ (\tilde{\gamma_1}\lambda_{\alpha_2})^{\lambda_\beta/p}].$$

$$\Big[\int_a^x p(s)\left(|D_a^\beta f_1(s)|^p+|D_a^\beta f_2(s)|^p\right)ds\Big]^{\frac{\left(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta\right)}{p}}. \tag{110}$$

Proof
Similar transfer to $a\in\Re$ of **Theorem 6** of [2]. ■

We proceed with a special important case.

Theorem 47 Let

$$\beta > \alpha_1 + 1,\ \alpha_1 \in \Re_+$$

and let

$$f_1, f_2 \in L_1(a,x),\ a,\ x \in \Re,\ a < x$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[0,x]$, and let

$$D_a^{\beta-k} f_i(a) = 0,\ for\ k = 1, \ldots, [\beta]+1;\ i = 1,2.$$

Consider also

$$p(t) > 0\ and\ q(t) \geq 0,$$

with

$$p(t),\ \frac{1}{p(t)}\ , q(t) \in L_\infty(a,x).$$

Let

$$\lambda_\alpha \geq 0, \quad 0 < \lambda_{\alpha+1} < 1,$$

and

$$p > 1.$$

Denote

$$\theta_3 := \begin{cases} 2^{\lambda_\alpha/\lambda_{\alpha+1}} - 1, & if \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ \\ 1, \quad if \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{111}$$

$$L(x-a) := \left(2\int_a^x (q(s))^{(1/(1-\lambda_{\alpha+1}))} ds\right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}}\right)^{\lambda_{\alpha+1}}, \tag{112}$$

and

$$P_1(x-a) := \int_a^x (x-s)^{(\beta-\alpha_1-1)p/(p-1)}\ (p(s))^{-1/(p-1)} ds, \tag{113}$$

$$T(x-a) := L(x-a)\left(\frac{P_1(x-a)^{\left(\frac{p-1}{p}\right)}}{\Gamma(\beta-\alpha_1)}\right)^{(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{114}$$

and

$$w_1 := 2^{\left(\frac{p-1}{p}\right)(\lambda_\alpha+\lambda_{\alpha+1})}, \tag{115}$$

with

$$\Phi(x-a) := T(x-a)w_1. \tag{116}$$

Then

$$\int_a^x q(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1} f_2(s)|^{\lambda_{\alpha+1}} + |D_a^{\alpha_1} f_2(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1} f_1(s)|^{\lambda_{\alpha+1}}\Big] ds$$

$$\leq \Phi(x-a)\Big[\int_a^x p(s)\left(|D_a^\beta f_1(s)|^p + |D_a^\beta f_2(s)|^p\right) ds\Big]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}. \tag{117}$$

Proof
Similar transfer to $a \in \Re$ of **Theorem 8** of [2]. ■

We give

Theorem 48 All here, as in **Theorem 44**. Consider the special case of

$$\lambda_{\alpha_2} = \lambda_{\alpha_1} + \lambda_\beta.$$

Denote

$$\tilde{T}(x-a) := A_0(x-a)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p} 2^{(p-2\lambda_{\alpha_1}-3\lambda_\beta)/p}. \tag{118}$$

Then, it holds

$$\int_a^x q(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^\beta f_1(s)|^{\lambda_\beta}$$

$$+|D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}}\ |D_a^\beta f_2(s)|^{\lambda_\beta}\Big] ds \leq$$

$$\tilde{T}\ (x-a)\Big(\int_a^x p(s)\left(|D_a^\beta f_1(s)|^p + |D_a^\beta f_2(s)|^p\right) ds\Big)^{\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}}. \tag{119}$$

Proof
Transfer of **Theorem 9** of [2]. ∎

Next, follow special cases of last theorems.

Corollary 49 (to **Theorem 44**) Set $\lambda_{\alpha_2} = 0$, $p(t) = q(t) = 1$.

Then

$$\int_a^x \left[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\beta} f_1(s)|^{\lambda_\beta} + |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}}\ |D_a^{\beta} f_2(s)|^{\lambda_\beta}\right] ds \leq$$

$$C_1(x-a)\Big[\int_a^x \left[|D_a^{\beta} f_1(s)|^p + |D_a^{\beta} f_2(s)|^p\right) ds\Big]^{\frac{(\lambda_{\alpha_1}+\lambda_\beta)}{p}}, \tag{120}$$

where

$$C_1(x-a) := \left(A_0(x-a)\Big|_{\lambda_{\alpha_2}=0}\right)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p} \delta_1, \tag{121}$$

$$\delta_1 := \begin{cases} 2^{1-\left((\lambda_{\alpha_1}+\lambda_\beta)/p\right)}, & if\ \ \lambda_{\alpha_1}+\lambda_\beta \leq p, \\ \\ 1, \ \ if\ \ \lambda_{\alpha_1}+\lambda_\beta \geq p. \end{cases} \tag{122}$$

We find that

$$\left(A_0(x-a)\Big|_{\lambda_{\alpha_2}=0}\right) = \Big\{\Big(\frac{(p-1)^{((\lambda_{\alpha_1}p-\lambda_{\alpha_1})/p)}}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}}(\beta p-\alpha_1 p-1)^{((\lambda_{\alpha_1}p-\lambda_{\alpha_1})/p)}}\Big)\times$$

$$\Big(\frac{(p-\lambda_\beta)^{((p-\lambda_\beta)/p)}}{(\lambda_{\alpha_1}\beta p-\lambda_{\alpha_1}\alpha_1 p-\lambda_{\alpha_1}+p-\lambda_\beta)^{((p-\lambda_\beta)/p)}}\Big)\Big\}\times(x-a)^{((\lambda_{\alpha_1}\beta p-\lambda_{\alpha_1}\alpha_1 p-\lambda_{\alpha_1}+p-\lambda_\beta)/p)}. \tag{123}$$

Proof
Transfer of **Corollary 10** of [2]. ∎

We continue with

Corollary 50 (to **Theorem 44**: Set $\lambda_{\alpha_2} = 0,\ p(t) = q(t) = 1,\ \lambda_{\alpha_1} = \lambda_\beta = 1,\ p = 2.$)

In detail, let

$$\alpha_1 \in \Re_+,\ \beta > \alpha_1,\ \beta - \alpha_1 > (1/2),$$

and let

$$f_1, f_2 \in L_1(a,x),\ a,\ x \in \Re,\ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a,x]$, and let

$$D_a^{\beta-k} f_i(a) = 0, \;\; for \;\; k = 1, \ldots, [\beta] + 1; \;\; i = 1, 2.$$

Then

$$\int_a^x \Big[|D_a^{\alpha_1} f_1(s)| \; |D_a^{\beta} f_1(s)| + |D_a^{\alpha_1} f_2(s)| \; |D_a^{\beta} f_2(s)| \Big] ds \leq$$

$$\left(\frac{(x-a)^{(\beta-\alpha_1)}}{2\Gamma(\beta-\alpha_1)\sqrt{\beta-\alpha_1}\;\sqrt{2\beta-2\alpha_1-1}} \right) \left(\int_a^x \Big[(D_a^{\beta} f_1(s))^2 + (D_a^{\beta} f_2(s))^2 \Big] ds \right). \tag{124}$$

Proof
Transfer of **Corollary 11** of [2]. ■

We continue with

Corollary 51 (to **Theorem 45**, $\lambda_{\alpha_1} = 0, \; p(t) = q(t) = 1.$) It holds

$$\int_a^x \Big[|D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_2}} \; |D_a^{\beta} f_1(s)|^{\lambda_\beta} + |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} \; |D_a^{\beta} f_2(s)|^{\lambda_\beta} \Big] ds \leq$$

$$C_2(x-a) \Big[\int_a^x \Big[|D_a^{\beta} f_1(s)|^p + |D_a^{\beta} f_2(s)|^p \Big) \, ds \Big]^{\frac{(\lambda_\beta + \lambda_{\alpha_2})}{p}}. \tag{125}$$

Here

$$C_2(x-a) := \left(A_0(x-a)\Big|_{\lambda_{\alpha_1}=0} \right) 2^{\left(\frac{p-\lambda_\beta}{p}\right)} \left(\frac{\lambda_\beta}{\lambda_{\alpha_2} + \lambda_\beta} \right)^{\lambda_\beta/p} \delta_3^{\lambda_\beta/p}, \tag{126}$$

$$\delta_3 := \begin{cases} 2^{\lambda_{\alpha_2}/\lambda_\beta} - 1, & if \;\; \lambda_{\alpha_2} \geq \lambda_\beta, \\ \\ 1, \;\; if \;\; \lambda_{\alpha_2} \leq \lambda_\beta. \end{cases} \tag{127}$$

We find that

$$\left(A_0(x-a)\Big|_{\lambda_{\alpha_1}=0} \right) = \Big\{ \Big(\frac{(p-1)^{(\lambda_{\alpha_2} p - \lambda_{\alpha_2})/p)}}{(\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_2}} (\beta p - \alpha_2 p - 1)^{(\lambda_{\alpha_2} p - \lambda_{\alpha_2})/p)}} \Big) \times$$

$$\Big(\frac{(p-\lambda_\beta)^{((p-\lambda_\beta)/p)}}{(\lambda_{\alpha_2}\beta p - \lambda_{\alpha_2}\alpha_2 p - \lambda_{\alpha_2} + p - \lambda_\beta)^{((p-\lambda_\beta)/p)}} \Big) \Big\} \times (x-a)^{((\lambda_{\alpha_2}\beta p - \lambda_{\alpha_2}\alpha_2 p - \lambda_{\alpha_2} + p - \lambda_\beta)/p)}. \tag{128}$$

Proof
Transfer of **Corollary 12** of [2]. ■

We give

Corollary 52 (to **Theorem 45**, $\lambda_{\alpha_1} = 0,\ p(t) = q(t) = 1,\ \lambda_{\alpha_2} = \lambda_\beta = 1,\ p = 2$.) In detail, let

$$\alpha_2 \in \Re_+,\ \beta > \alpha_2,\ \beta - \alpha_2 > (1/2),$$

and let

$$f_1, f_2 \in L_1(a, x),\ a,\ x \in \Re,\ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a, x]$, and let

$$D_a^{\beta-k} f_i(a) = 0,\ for\ k = 1, \ldots, [\beta] + 1;\ i = 1, 2.$$

Then, it holds

$$\int_a^x \left[|D_a^{\alpha_2} f_2(s)|\ |D_a^\beta f_1(s)| + |D_a^{\alpha_2} f_1(s)|\ |D_a^\beta f_2(s)| \right] ds \leq$$

$$C_2^*(x-a) \left(\int_a^x \left[(D_a^\beta f_1(s))^2 + (D_a^\beta f_2(s))^2 \right] ds \right), \tag{129}$$

where

$$C_2^*(x-a) := \frac{(x-a)^{(\beta-\alpha_2)}}{\sqrt{2}\ \Gamma(\beta-\alpha_2)\sqrt{\beta-\alpha_2}\ \sqrt{2\beta-2\alpha_2-1}}. \tag{130}$$

Proof

Transfer of **Corollary 13** of [2]. ■

We continue with

Corollary 53 (to **Theorem 46**, $\lambda_{\alpha_1} = \lambda_{\alpha_2} = \lambda_\beta = 1,\ p = 3,\ p(t) = q(t) = 1$.) It holds

$$\int_a^x \left[|D_a^{\alpha_1} f_1(s)|\ |D_a^{\alpha_2} f_2(s)|\ |D_a^\beta f_1(s)| + |\ D_a^{\alpha_2} f_1(s)|\ |D_a^{\alpha_1} f_2(s)|\ |D_a^\beta f_2(s)| \right] ds \leq$$

$$A_0(x-a) \left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{6}} \right) \left(\int_a^x \left(|D_a^\beta f_1(s)|^3 + |D_a^\beta f_2(s)|^3 \right) ds \right) \tag{131}$$

Here,

$$A_0(x-a) = 4(x-a)^{(2\beta-\alpha_1-\alpha_2)} \times$$

$$\frac{1}{\Gamma(\beta-\alpha_1)\ \Gamma(\beta-\alpha_2)[3(3\beta-3\alpha_1-1)(3\beta-3\alpha_2-1)(2\beta-\alpha_1-\alpha_2)]^{2/3}} \tag{132}$$

Proof
Transfer of **Corollary 14** of [2]. ■

We give

Corollary 54 (to **Theorem 47**, here $\lambda_\alpha = 1,\ \lambda_{\alpha+1} = 1/2\ p = 3/2,\ p(t) = q(t) = 1$.) In detail: let

$$\beta > \alpha_1 + 1,\ \alpha_1 \in \Re_+,$$

and let

$$f_1, f_2 \in L_1(a,x),\ a,\ x \in \Re,\ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a,x]$, and let

$$D_a^{\beta-k} f_i(a) = 0,\ for\ k = 1, \ldots, [\beta]+1;\ i = 1,2.$$

Then, it holds

$$\int_a^x \left[|D_a^{\alpha_1} f_1(s)|\ \sqrt{|D_a^{\alpha_1+1} f_2(s)|} + |D_a^{\alpha_1} f_2(s)|\ \sqrt{|D_a^{\alpha_1+1} f_1(s)|} \right] ds \leq$$

$$\Phi^*(x-a) \left[\int_a^x \left(|D_a^\beta f_1(s)|^{3/2} + |D_a^\beta f_2(s)|^{3/2} \right) ds \right], \tag{133}$$

where

$$\Phi^*(x-a) := \frac{2(x-a)^{(3\beta-3\alpha_1-1)/2}}{(\Gamma(\beta-\alpha_1))^{3/2}\ \sqrt{3\beta-3\alpha_1-2}}. \tag{134}$$

Proof
Transfer of **Corollary 15** of [2]. ■

We continue

Corollary 55 (to **Theorem 48**, here $p = 2(\lambda_{\alpha_1} + \lambda_\beta) > 1,\ p(t) = q(t) = 1$.) It holds

$$\int_a^x \Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^\beta f_1(s)|^{\lambda_\beta}$$

$$+ |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}}\ |D_a^\beta f_2(s)|^{\lambda_\beta} \Big] ds \leq$$

$$\tilde{\tilde{T}}(x-a) \Big(\int_a^x \left(|D_a^\beta f_1(s)|^{2(\lambda_{\alpha_1}+\lambda_\beta)} + |D_a^\beta f_2(s)|^{2(\lambda_{\alpha_1}+\lambda_\beta)} \right) ds \Big). \tag{135}$$

Here,

$$\tilde{\tilde{T}}(x-a) := \tilde{A}_0(x-a)\left(\frac{\lambda_\beta}{2(\lambda_{\alpha_1}+\lambda_\beta)}\right)^{\left(\frac{\lambda_\beta}{2(\lambda_{\alpha_1}+\lambda_\beta)}\right)}, \tag{136}$$

and

$$\tilde{A}_0(x-a) := \theta\theta^*(x-a)^{\theta}1, \tag{137}$$

where

$$\theta := \frac{1}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_1}+\lambda_\beta}}$$

$$\cdot\left(\frac{2\lambda_{\alpha_1}+2\lambda_\beta-1}{2\lambda_{\alpha_1}\beta+2\lambda_\beta\beta-2\lambda_{\alpha_1}\alpha_1-2\lambda_\beta\alpha_1-1}\right)^{\frac{(2\lambda_{\alpha_1}^2+2\lambda_{\alpha_1}\lambda_\beta-\lambda_{\alpha_1})}{(2\lambda_{\alpha_1}+2\lambda_\beta)}}$$

$$\cdot\left(\frac{2\lambda_{\alpha_1}+2\lambda_\beta-1}{2\lambda_{\alpha_1}\beta+2\lambda_\beta\beta-2\lambda_{\alpha_1}\alpha_2-2\lambda_\beta\alpha_2-1}\right)^{((2\lambda_{\alpha_1}+2\lambda_\beta-1)/2)}, \tag{138}$$

$$\theta^* := \left(\frac{2\lambda_{\alpha_1}+\lambda_\beta}{S}\right)^{\frac{(2\lambda_{\alpha_1}+\lambda_\beta)}{2(\lambda_{\alpha_1}+\lambda_\beta)}},$$

where

$$S := 4\lambda_{\alpha_1}^2\beta+6\lambda_{\alpha_1}\lambda_\beta\beta-2\lambda_{\alpha_1}^2\alpha_1-2\lambda_{\alpha_1}\lambda_\beta\alpha_1-2\lambda_{\alpha_1}^2\alpha_2-4\lambda_{\alpha_1}\lambda_\beta\alpha_2+2{\lambda_\beta}^2\beta-2{\lambda_\beta}^2\alpha_2$$

and

$$\theta_1 := \left(\frac{S}{2\lambda_{\alpha_1}+2\lambda_\beta}\right). \tag{139}$$

Proof
Transfer of **Corollary 16** of [2]. ■

We give the interesting special case

Corollary 56 (to **Theorem 48**, here $p=4,\ \lambda_{\alpha_1}=\lambda_\beta=1,\ p(t)=q(t)=1$.) It holds

$$\int_a^x\left[\left|D_a^{\alpha_1}f_1(s)\right|\left(D_a^{\alpha_2}f_2(s)\right)^2\left|D_a^{\beta}f_1(s)\right|+\left(D_a^{\alpha_2}f_1(s)\right)^2\left|D_a^{\alpha_1}f_2(s)\right|\left|D_a^{\beta}f_2(s)\right|\right]ds \le$$

$$T^*(x-a)\left(\int_a^x\left(D_a^{\beta}f_1(s))^4+(D_a^{\beta}f_2(s)^4\right)ds\right). \tag{140}$$

Here,

$$T^*(x-a) := \frac{A_0^*(x-a)}{\sqrt{2}}, \tag{141}$$

and

$$A^*(x-a) := \tilde{\theta}\tilde{\theta}^*(x-a)^{\tilde{\theta}1}, \tag{142}$$

where

$$\tilde{\theta} := \frac{1}{\Gamma(\beta-\alpha_1)(\Gamma(\beta-\alpha_2))^2}\left(\frac{3}{4\beta-4\alpha_1-1}\right)^{3/4}\left(\frac{3}{4\beta-4\alpha_2-1}\right)^{3/2}, \tag{143}$$

$$\tilde{\theta}^* := \left(\frac{3}{12\beta-4\alpha_1-8\alpha_2}\right)^{3/4}, \tag{144}$$

and

$$\tilde{\theta_1} := 3\beta-\alpha_1-2\alpha_2. \tag{145}$$

Proof
Transfer of **Corollary 17** of [2]. ■

We continue with related results regarding the L_∞- norm $\|\cdot\|_\infty$.

Theorem 57 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2$$

and let

$$f_1, f_2 \in L_1(a,x),\ a,\ x \in \Re,\ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a,x]$, and let

$$D_a^{\beta-k} f_i(a) = 0,\ for\ k = 1, \ldots, [\beta]+1;\ i = 1, 2.$$

Consider also

$$p(s) \geq 0,\ p(s) \in L_\infty(a,x).$$

Let $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \lambda_\beta \geq 0$. Set

$$\rho(x-a) := \frac{\|p(s)\|_\infty \ (x-a)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}} (\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}} [\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1]}. \tag{146}$$

Then

$$\int_a^x p(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}} \ |D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_2}} \ |D_a^{\beta} f_1(s)|^{\lambda_\beta}$$

$$+| \ D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} \ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}} \ |D_a^{\beta} f_2(s)|^{\lambda_\beta}\Big] ds \leq$$

$$\frac{\rho(x-a)}{2}\Big[\|D_a^\beta f_1\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)} + \|D_a^\beta f_1\|_\infty^{2\lambda_{\alpha_2}} + \|D_a^\beta f_2\|_\infty^{2\lambda_{\alpha_2}} + \|D_a^\beta f_2\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)}\Big]. \tag{147}$$

Proof

Transfer of **Theorem 18** of [2]. Similar to the proof of **Theorem 24**, etc. ■

We give special cases of last theorem.

Theorem 58 (all as in **Theorem 57**; $\lambda_{\alpha_2} = 0$.) It holds

$$\int_a^x p(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}} \ |D_a^{\beta} f_1(s)|^{\lambda_\beta} + |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}} \ |D_a^{\beta} f_2(s)|^{\lambda_\beta}\Big] ds \leq$$

$$\frac{\|p(s)\|_\infty \ (x-a)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + 1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}} [\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + 1]} \cdot \Big[\|D_a^\beta f_1\|_\infty^{\lambda_{\alpha_1}+\lambda_\beta} + \|D_a^\beta f_2\|_\infty^{\lambda_{\alpha_1}+\lambda_\beta}\Big]. \tag{148}$$

Proof

Transfer of **Theorem 19** of [2]. Similar to the proof of **Theorem 57**. ■

We continue with

Theorem 59 (all as in **Theorem 57**; $\lambda_{\alpha_2} = \lambda_{\alpha_1} + \lambda_\beta$.) It holds

$$\int_a^x p(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}} \ |D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_1}+\lambda_\beta} \ |D_a^{\beta} f_1(s)|^{\lambda_\beta}$$

$$+| \ D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta} \ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}} \ |D_a^{\beta} f_2(s)|^{\lambda_\beta}\Big] ds$$

$$\leq \Big\{ \frac{\|p(s)\|_\infty}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}} (\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}+\lambda_\beta}}$$

$$\cdot \frac{(x-a)^{(2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_\beta - \alpha_2\lambda_{\alpha_1} - \alpha_2\lambda_\beta + 1)}}{(2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_\beta - \alpha_2\lambda_{\alpha_1} - \alpha_2\lambda_\beta + 1)}\Big\} \cdot \Big[\|D_a^\beta f_1\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)} + \|D_a^\beta f_2\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)}\Big]. \tag{149}$$

Proof
Transfer of **Theorem 20** of [2]. ■

We give

Theorem 60 (all as in **Theorem 57**; $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2}$.) It holds

$$\int_a^x p(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_1}} + |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}}\Big]ds \leq$$

$$\rho^*(x-a)\Big[\|D_a^\beta f_1\|_\infty^{2\lambda_{\alpha_1}} + \|D_a^\beta f_2\|_\infty^{2\lambda_{\alpha_1}}\Big], \tag{150}$$

where

$$\rho^*(x-a) = \left\{\frac{\|p(s)\|_\infty\ (x-a)^{(2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} - \alpha_2\lambda_{\alpha_1} + 1)}}{(\Gamma(\beta-\alpha_1+1)\ \Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}}\ (2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} - \alpha_2\lambda_{\alpha_1} + 1)}\right\}. \tag{151}$$

Proof
Transfer of **Theorem 21** of [2]. ■

We give

Theorem 61 (all as in **Theorem 57**; $\lambda_{\alpha_1} = 0,\ \lambda_{\alpha_2} = \lambda_\beta$.) It holds

$$\int_a^x p(s)\Big[|D_a^{\alpha_2} f_2(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_1(s)|^{\lambda_{\alpha_2}} + |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_2(s)|^{\lambda_{\alpha_2}}\Big]ds \leq$$

$$\left(\frac{(x-a)^{(\beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}\ \|p(s)\|_\infty}{(\beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}}\right)\cdot\Big[\|D_a^\beta f_1\|_\infty^{2\lambda_{\alpha_2}} + \|D_a^\beta f_2\|_\infty^{2\lambda_{\alpha_2}}\Big]. \tag{152}$$

Proof
Transfer of **Theorem 22** of [2]. ■

We continue with

Corolarry 62 (to **Theorem 60**, all as in **Theorem 57**; $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2},\ \alpha_2 = \alpha_1 + 1$.) It holds

$$\int_a^x p(s)\Big[|D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1+1} f_2(s)|^{\lambda_{\alpha_1}} + |D_a^{\alpha_1+1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1} f_2(s)|^{\lambda_{\alpha_1}}\Big]ds \leq$$

$$\left(\frac{(x-a)^{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)}\ \|p(s)\|_\infty}{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)(\beta-\alpha_1)^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_1))^{2\lambda_{\alpha_1}}}\right)\cdot\Big[\|D_a^\beta f_1\|_\infty^{2\lambda_{\alpha_1}} + \|D_a^\beta f_2\|_\infty^{2\lambda_{\alpha_1}}\Big]. \tag{153}$$

Proof
Transfer of **Corolarry 23** of [2]. ■

We give

Corolarry 63 (to **Corolarry 62**) In detail: let

$$\alpha_1 \in \Re_+, \ \beta > \alpha_1 + 1,$$

and let

$$f_1, f_2 \in L_1(a, x), \ a, \ x \in \Re, \ a < x$$

have respectively L_∞ fractional derivatives $D_a^\beta f_1, D_a^\beta f_2$ in $[a, x]$, and let

$$D_a^{\beta-k} f_i(a) = 0, \ for \ k = 1, \ldots, [\beta] + 1; \ i = 1, 2.$$

Then, it holds

$$\int_a^x \Big[|D_a^{\alpha_1} f_1(s)| \ |D_a^{\alpha_1+1} f_2(s)| + |D_a^{\alpha_1+1} f_1(s)| \ |D_a^{\alpha_1} f_2(s)| \Big] ds \leq$$

$$\left(\frac{(x-a)^{2(\beta-\alpha_1)}}{2(\beta-\alpha_1)^2 (\Gamma(\beta-\alpha_1))^2} \right) \cdot \Big[\|D_a^\beta f_1\|_\infty^2 + \|D_a^\beta f_2\|_\infty^2 \Big]. \qquad (154)$$

Proof

Transfer of **Corolarry 24** of [2]. ■

We finally give

Proposition 64 Inequality (154) is sharp, infact it is attained when

$$f_1 = f_2,$$

by

$$f_1(s) = (s-a)^\beta, \ a \leq s \leq x, \ \beta > \alpha_1 + 1, \ \alpha_1 \geq 0.$$

Proof

Clearly (154) when $f_1 = f_2$, collapses to

$$\int_a^x |D_a^{\alpha_1} f_1(s)| \ |D_a^{\alpha_1+1} f_1(s)| ds \leq \left(\frac{(x-a)^{2(\beta-\alpha_1)}}{2(\Gamma(\beta-\alpha_1+1))^2} \right) \cdot \|D_a^\beta f_1\|_\infty^2, \quad (155)$$

see **Theorem 27** and **Proposition 28**. ■

Next we apply above results on the spherical shell A.

We make

Assumption 65 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2,\ \beta - \alpha_i > (1/p), p > 1, i = 1, 2,$$

and let

$$f_1, f_2 \in L_1(A)$$

with

$$\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}},\ \frac{\partial^{\beta}_{R_1} f_2(x)}{\partial r^{\beta}} \in L_\infty(A),\ x \in A;$$

$$A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \Re^N,\ N \geq 2,\ 0 < R_1 < R_2.$$

Further assume that each $D^{\beta}_{R_1} f_i(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D^{\beta}_{R_1} f_i(rw)| \leq M_i$$

for some $M_i > 0;\ i = 1, 2$. For each $w \in S^{N-1} - (K(f_1) \cup K(f_2))$, we assume that $f_i(\cdot\ w)$ has an L_∞ fractional derivative $D^{\beta}_{R_1} f_i(\cdot\ w)$ in $[R_1, R_2]$, and that

$$D^{\beta-k}_{R_1} f_i(R_1 w) = 0,\ k = 1, \ldots, [\beta] + 1;$$

$i = 1, 2$. Let $\lambda_\beta > 0$ and $\lambda_{\alpha_1},\ \lambda_{\alpha_2} \geq 0$, such that $\lambda_\beta < p$. If $\alpha_1 = 0$ we set $\lambda_{\alpha_1} = 1$, and if $\alpha_2 = 0$ we set $\lambda_{\alpha_2} = 1$.

We need

Notation 66 (on **Assumption 65**) Set

$$P_i(s) := \int_0^s (s - r)^{\frac{p(\beta - \alpha_i - 1)}{p-1}} (r + R_1)^{\frac{1-N}{p-1}} dr,\ i = 1, 2;\ 0 \leq s \leq R_2 - R_1, \tag{156}$$

$$A(s) := \frac{(s + R_1)^{(N-1)\left(1 - \left(\frac{\lambda_\beta}{p}\right)\right)} (P_1(s))^{\lambda_{\alpha_1}\left(\frac{p-1}{p}\right)}\ (P_2(s))^{\lambda_{\alpha_2}\left(\frac{p-1}{p}\right)}}{(\Gamma(\beta - \alpha_1))^{\lambda_{\alpha_1}}\ (\Gamma(\beta - \alpha_2))^{\lambda_{\alpha_2}}}, \tag{157}$$

and

$$A_0(R_2 - R_1) := \left(\int_0^{R_2 - R_1} (A(s))^{p/(p-\lambda_\beta)} ds\right)^{(p-\lambda_\beta)/p}. \tag{158}$$

We present

Theorem 67 (All as in **Assumption 65** and **Notation 66**). Here $\lambda_{\alpha_1} > 0$, $\lambda_{\alpha_2} = 0$ and $p = \lambda_{\alpha_1} + \lambda_\beta > 1$. Then

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} + \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$\left(A_0(R_2 - R_1)\Big|_{\lambda_{\alpha_2}=0} \right) \left(\frac{\lambda_\beta}{\lambda_{\alpha_1} + \lambda_\beta} \right)^{\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta} \right)} \left[\int_A \Big[\Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{p} + \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{p} \Big] dx \right]. \tag{159}$$

Proof

By **Theorem 20** for $\alpha_1 > 0$ we get that

$$\frac{\partial_{R_1}^{\alpha_1} f_i(x)}{\partial r^{\alpha_1}} \in L\infty(A), \ i = 1, 2.$$

In general here the integrands of both integrals of (159) are in $L_1(A)$. Thus, by **Proposition 3** we have

$$I_1 := L.H.S(159) = \int_{S^{N-1}} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\alpha_1} f_1(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\beta} f_1(rw)|^{\lambda_\beta} +$$

$$|D_{R_1}^{\alpha_1} f_2(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\beta} f_2(rw)|^{\lambda_\beta}] r^{N-1} dr \Big) dw =$$

$$\int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\alpha_1} f_1(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\beta} f_1(rw)|^{\lambda_\beta} +$$

$$|D_{R_1}^{\alpha_1} f_2(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\beta} f_2(rw)|^{\lambda_\beta}] r^{N-1} dr \Big) dw. \tag{160}$$

Similary we have

$$I_2 := \int_A \Big[\Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{p} + \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{p} \Big] dx$$

$$= \int_{S^{N-1}} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\beta} f_1(rw)|^p + |D_{R_1}^{\beta} f_2(rw)|^p \,] r^{N-1} dr \Big) dw =$$

$$\int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\beta} f_1(rw)|^p + |D_{R_1}^{\beta} f_2(rw)|^p \,] r^{N-1} dr \Big) dw. \tag{161}$$

Notice here $\lambda_{S^{N-1}}(K(f_1) \cup K(f_2)) = 0$.

Here for every $w \in S^{N-1} - (K(f_1) \cup K(f_2))$ and for $p(r) = q(r) = r^{N-1},\ r \in [R_1, R_2],\ N \geq 2$ we apply **Theorem 44**. We obtain

$$\int_{R_1}^{R_2} [\,|D_{R_1}^{\alpha_1} f_1(rw)|^{\lambda_{\alpha_1}}\ |D_{R_1}^{\beta} f_1(rw)|^{\lambda_\beta} + |D_{R_1}^{\alpha_1} f_2(rw)|^{\lambda_{\alpha_1}}\ |D_{R_1}^{\beta} f_2(rw)|^{\lambda_\beta}] r^{N-1} dr \leq$$

$$\left(A_0(R_2 - R_1)\Big|_{\lambda_{\alpha_2}=0} \right) \left(\frac{\lambda_\beta}{\lambda_{\alpha_1} + \lambda_\beta} \right)^{\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)} \times$$

$$\left(\int_{R_1}^{R_2} [\,|D_{R_1}^{\beta} f_1(rw)|^p + |D_{R_1}^{\beta} f_2(rw)|^p\,] r^{N-1} \right) dr \tag{162}$$

Integrating now (162) over $S^{N-1} - (K(f_1) \cup K(f_2))$ and taking into account (160) and (161) we derive (159). ■

We continue with

Theorem 68 (All as in **Assumption 65** and **Notation 66**). Here $\lambda_{\alpha_1} = 0,\ \lambda_{\alpha_2} > 0$ and $p = \lambda_\beta + \lambda_{\alpha_2} > 1$. Denote

$$\delta_3 := \begin{cases} 2^{\lambda_{\alpha_2}/\lambda_\beta} - 1, & if \quad \lambda_{\alpha_2} \geq \lambda_\beta, \\ 1, & if \quad \lambda_{\alpha_2} \leq \lambda_\beta. \end{cases} \tag{163}$$

Then

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$\left(A_0(R_2 - R_1)\Big|_{\lambda_{\alpha_1}=0} \right) 2^{\lambda_{\alpha_2}/p} \left(\frac{\lambda_\beta}{p} \right)^{\lambda_\beta/p} \delta_3^{\lambda_\beta/p} \left(\int_A \Big[\Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^p + \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^p \Big] dx \right). \tag{164}$$

Proof

Based on **Theorem 45**, similar to the proof of **Theorem 67**. ■

The complete case $\lambda_{\alpha_1},\ \lambda_{\alpha_2} > 0$ follows.

Theorem 69 (All as in **Assumption 65** and **Notation 66**). Here $\lambda_{\alpha_1},\ \lambda_{\alpha_2} > 0$, $p = \lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta > 1$. Denote

$$\tilde{\gamma_1} := \begin{cases} 2^{((\lambda_{\alpha_1}+\lambda_{\alpha_2})/\lambda_\beta)} - 1, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} \geq \lambda_\beta, \\ 1, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} \leq \lambda_\beta, \end{cases} \tag{165}$$

Then

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$A_0(R_2 - R_1) \left(\frac{\lambda_\beta}{(\lambda_{\alpha_1} + \lambda_{\alpha_2}) p} \right)^{(\lambda_\beta / p)} [\lambda_{\alpha_1}{}^{(\lambda_\beta/p)} + 2^{(p-\lambda_\beta)/p} \, (\tilde{\gamma}_1 \lambda_{\alpha_2})^{(\lambda_\beta/p)}] \cdot$$

$$\left(\int_A \Big[\Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^p + \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^p \Big] dx \right). \tag{166}$$

Proof
Based on **Theorem 46**, similar to the proof **Theorem 67**. ■

A special important case it next.

Theorem 70 All as in **Assumption 65**. Here $\alpha_2 = \alpha_1 + 1$, $\lambda_\alpha := \lambda_{\alpha_1} \geq 0$, $\lambda_{\alpha+1} := \lambda_{\alpha_2} \in (0,1)$, and $p = \lambda_\alpha + \lambda_{\alpha+1} > 1$. Denote

$$\theta_3 := \begin{cases} 2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1, & if \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ 1, & if \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{167}$$

$$L(R_2 - R_1) := \left[2 \, \frac{(1-\lambda_{\alpha+1})}{(N-\lambda_{\alpha+1})} \left(R_2^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} - R_1^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} \right) \right]^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \, \lambda_{\alpha+1}}{p} \right)^{\lambda_{\alpha+1}}, \tag{168}$$

and

$$P_1(R_2 - R_1) := \int_{R_1}^{R_2} (R_2 - r)^{(\beta - \alpha_1 - 1)p/(p-1)} \, r^{(1-N)/(p-1)} dr, \tag{169}$$

$$\Phi(R_2 - R_1) := L(R_2 - R_1) \left(\frac{(P_1(R_2 - R_1))^{(p-1)}}{(\Gamma(\beta - \alpha_1))^p} \right) 2^{(p-1)}. \tag{170}$$

Then

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_\alpha} \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_2(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha+1}} + \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_\alpha} \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_1(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha+1}} \Big] dx \leq$$

$$\Phi(R_2-R_1)\left(\int_A \Big[\Big|\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}}\Big|^p + \Big|\frac{\partial^{\beta}_{R_1} f_2(x)}{\partial r^{\beta}}\Big|^p\Big]dx\right). \qquad (171)$$

Proof
Based on **Theorem 47**, similar to the proof of **Theorem 67**. ■

We also give

Theorem 71 (All as in **Assumption 65** and **Notation 66**). Here $\lambda_{\alpha_2} = \lambda_{\alpha_1} + \lambda_\beta$ and $p = 2(\lambda_{\alpha_1} + \lambda_\beta) > 1$. Denote

$$\tilde{T}(R_2-R_1) := A_0(R_2-R_1)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p} 2^{-\lambda_\beta/p}. \qquad (172)$$

Then

$$\int_A \Big[\Big|\frac{\partial^{\alpha_1}_{R_1} f_1(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}} \Big|\frac{\partial^{\alpha_2}_{R_1} f_2(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta} \Big|\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta} +$$

$$\Big|\frac{\partial^{\alpha_2}_{R_1} f_1(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta} \Big|\frac{\partial^{\alpha_1}_{R_1} f_2(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}} \Big|\frac{\partial^{\beta}_{R_1} f_2(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]dx \le$$

$$\tilde{T}(R_2-R_1)\left(\int_A \Big[\Big|\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}}\Big|^p + \Big|\frac{\partial^{\beta}_{R_1} f_2(x)}{\partial r^{\beta}}\Big|^p\Big]dx\right). \qquad (173)$$

Proof
Based on **Theorem 48**. ■

We continue with related L_∞ results on the shell A. We make

Assumption 72 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2$$

and let

$$f_1, f_2 \in L_1(A)$$

with

$$\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}},\ \frac{\partial^{\beta}_{R_1} f_2(x)}{\partial r^{\beta}} \in L_\infty(A),\ x \in A;$$

$$A := B(0,R_2) - \overline{B(0,R_1)} \subseteq \Re^N,\ N \ge 2,\ 0 < R_1 < R_2.$$

Further assume that each

$D^{\beta}_{R_1} f_i(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D_{R_1}^{\beta} f_i(rw)| \leq M_i$$

for some $M_i > 0$; $i = 1, 2$. For each $w \in S^{N-1} - (K(f_1) \cup K(f_2))$, we assume that $f_i(\cdot\ w)$ has an L_∞ fractional derivative $D_{R_1}^{\beta} f_i(\cdot\ w)$ in $[R_1, R_2]$, and that

$$D_{R_1}^{\beta-k} f_i(R_1 w) = 0,\ k = 1, \ldots, [\beta] + 1;$$

$i = 1, 2$. Let $\lambda_{\alpha_1},\ \lambda_{\alpha_2},\ \lambda_\beta \geq 0$. If $\alpha_1 = 0$ we set $\lambda_{\alpha_1} = 1$, and if $\alpha_2 = 0$ we set $\lambda_{\alpha_2} = 1$.

We present

Theorem 73 All here as in **Assumption 72**. Set

$$\rho(R_2 - R_1) = \frac{R_2^{N-1}(R_2 - R_1)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\Gamma(\beta - \alpha_1 + 1))^{\lambda_{\alpha_1}} (\Gamma(\beta - \alpha_2 + 1))^{\lambda_{\alpha_2}} (\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)} \quad (174)$$

Then

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$\rho(R_2 - R_1)[{M_1}^{2(\lambda_{\alpha_1} + \lambda_\beta)} + {M_1}^{2\lambda_{\alpha_2}} + {M_2}^{2\lambda_{\alpha_2}} + {M_2}^{2(\lambda_{\alpha_1} + \lambda_\beta)}] \frac{\pi^{N/2}}{\Gamma(N/2)}. \quad (175)$$

Proof

It is based on **Theorem 57**. By **Theorem 20** for $\alpha_j > 0$ we get that

$$\frac{\partial_{R_1}^{\alpha_j} f_i}{\partial r^{\alpha_j}} \in L_\infty(A),\ i = 1, 2;\ j = 1, 2.$$

In general here the integrand of the integral of (175) belongs to $L_1(A)$. Thus, by **Proposition 3** we have

$$L.H.S(175) = \int_{S^{N-1}} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\alpha_1} f_1(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\alpha_2} f_2(rw)|^{\lambda_{\alpha_2}} \, |D_{R_1}^{\beta} f_1(rw)|^{\lambda_\beta} +$$

$$|D_{R_1}^{\alpha_2} f_1(rw)|^{\lambda_{\alpha_2}} \, |D_{R_1}^{\alpha_1} f_2(rw)|^{\lambda_{\alpha_1}} \, |D_{R_1}^{\beta} f_2(rw)|^{\lambda_\beta}] r^{N-1} dr \Big) dw \quad (176)$$

$$= \int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} \Big(\int_{R_1}^{R_2} [\, |D_{R_1}^{\alpha_1} f_1(rw)|^{\lambda_{\alpha_1}} \ |D_{R_1}^{\alpha_2} f_2(rw)|^{\lambda_{\alpha_2}} \ |D_{R_1}^{\beta} f_1(rw)|^{\lambda_\beta} +$$

$$|D_{R_1}^{\alpha_2} f_1(rw)|^{\lambda_{\alpha_2}} \ |D_{R_1}^{\alpha_1} f_2(rw)|^{\lambda_{\alpha_1}} \ |D_{R_1}^{\beta} f_2(rw)|^{\lambda_\beta}] r^{N-1} dr \Big) dw \tag{177}$$

(by (147) for $p(r) = r^{N-1}$, $r \in [R_1, R_2]$)

$$\leq \frac{\rho(R_2 - R_1)}{2} \int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} \Big[\|D_{R_1}^{\beta} f_1(\cdot\ w)\|_{\infty,\ [R_1,R_2]}^{2(\lambda_{\alpha_1}+\lambda_\beta)} +$$

$$\|D_{R_1}^{\beta} f_1(\cdot\ w)\|_{\infty,\ [R_1,R_2]}^{2\lambda_{\alpha_2}} + \|D_{R_1}^{\beta} f_2(\cdot\ w)\|_{\infty,\ [R_1,R_2]}^{2\lambda_{\alpha_2}} + \|D_{R_1}^{\beta} f_2(\cdot\ w)\|_{\infty,\ [R_1,R_2]}^{2(\lambda_{\alpha_1}+\lambda_\beta)} \Big] dw \tag{178}$$

$$\leq \frac{\rho(R_2 - R_1)}{2} \Big[{M_1}^{2(\lambda_{\alpha_1}+\lambda_\beta)} + {M_1}^{2\lambda_{\alpha_2}} + {M_2}^{2\lambda_{\alpha_2}} + {M_2}^{2(\lambda_{\alpha_1}+\lambda_\beta)} \Big]. \tag{179}$$

$$\int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} dw =$$

$$\frac{\rho(R_2 - R_1)}{2} \Big[{M_1}^{2(\lambda_{\alpha_1}+\lambda_\beta)} + {M_1}^{2\lambda_{\alpha_2}} + {M_2}^{2\lambda_{\alpha_2}} + {M_2}^{2(\lambda_{\alpha_1}+\lambda_\beta)} \Big] \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{180}$$

by

$$\int_{(S^{N-1}-(K(f_1)\cup K(f_2)))} dw = \int_{S^{N-1}} dw = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \tag{181}$$

since $\lambda_{S^{N-1}}(K(f_1) \cup K(f_2)) = 0$. That is proving inequality (175). ■

We give special cases of last theorem.

Theorem 74 (All here as in **Assumption 72**, here $\lambda_{\alpha_2} = 0$.) It holds

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} + \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$\frac{{R_2}^{N-1} (R_2 - R_1)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + 1)}}{(\Gamma(\beta - \alpha_1 + 1))^{\lambda_{\alpha_1}} \ [\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + 1]} \times$$

$$\Big[{M_1}^{(\lambda_{\alpha_1}+\lambda_\beta)} + {M_2}^{(\lambda_{\alpha_1}+\lambda_\beta)} \Big] \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{182}$$

Proof

Based on **Theorem 58** and similar to the proof of **Theorem 73**. ■

We continue with

Theorem 75 (All here as in **Assumption 72**, here $\lambda_{\alpha_2} = \lambda_{\alpha_1} + \lambda_\beta$.)
It holds

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}+\lambda_\beta} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}+\lambda_\beta} \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] dx \leq$$

$$\frac{{R_2}^{N-1}(R_2 - R_1)^{(2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_\beta - \alpha_2\lambda_{\alpha_1} - \alpha_2\lambda_\beta + 1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}+\lambda_\beta}\ [2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_\beta - \alpha_2\lambda_{\alpha_1} - \alpha_2\lambda_\beta + 1]}$$

$$\times \Big[{M_1}^{2(\lambda_{\alpha_1}+\lambda_\beta)} + {M_2}^{2(\lambda_{\alpha_1}+\lambda_\beta)} \Big] \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{183}$$

Proof
Based on **Theorem 59** and similar to the proof of **Theorem 73**. ■

We give

Theorem 76 (All here as in **Assumption 72**, here $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2}$).
It holds

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] dx \leq$$

$$\frac{{R_2}^{N-1}(R_2 - R_1)^{(2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} - \alpha_2\lambda_{\alpha_1} + 1)}}{(\Gamma(\beta-\alpha_1+1)\ \Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}}\ (2\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} - \alpha_2\lambda_{\alpha_1} + 1)} \times$$

$$\Big[{M_1}^{2\lambda_{\alpha_1}} + {M_2}^{2\lambda_{\alpha_1}} \Big] \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{184}$$

Proof
Based on **Theorem 60** and similar to the proof of **Theorem 73**. ■

We give

Theorem 77 (All here as in **Assumption 72**, here $\lambda_{\alpha_1} = 0,\ \lambda_{\alpha_2} = \lambda_\beta$.)
It holds

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_2} f_2(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_2(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} \Big] dx \leq$$

$$\frac{{R_2}^{N-1}(R_2 - R_1)^{(\beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}} \Big({M_1}^{2\lambda_{\alpha_2}} + {M_2}^{2\lambda_{\alpha_2}} \Big) \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{185}$$

Proof
Based on **Theorem 61**, etc. ■

We finish this section with

Corollary 78 (to **Theorem 76**. All here as in **Assumption 72**, $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2},\ \alpha_2 = \alpha_1 + 1$). It holds

$$\int_A \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_2(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_1(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial_{R_1}^{\alpha_1} f_2(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] dx \le$$

$$\frac{{R_2}^{N-1} (R_2 - R_1)^{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)}}{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)(\beta - \alpha_1)^{\lambda_{\alpha_1}} \ (\Gamma(\beta - \alpha_1))^{2\lambda_{\alpha_1}}} \times$$

$$\Big[{M_1}^{2\lambda_{\alpha_1}} + {M_2}^{2\lambda_{\alpha_1}} \Big] \frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{186}$$

Proof

Based on **Corollary 62**, etc. ■

5.3 Riemann- Liouiville fractional Opial type inequalities involving several functions

We present

Theorem 79 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2,\ \beta - \alpha_i > (1/p), p > 1, i = 1, 2,$$

and let

$$f_j \in L_1(a, x),\ j = 1, \ldots, M \in \mathcal{N},\ a,\ x \in \Re,\ a < x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a, x]$, and let

$$D_a^{\beta-k} f_j(a) = 0,\ for\ k = 1, \ldots, [\beta] + 1;\ j = 1, \ldots, M.$$

Consider also

$$p(t) > 0\ and\ q(t) \ge 0,$$

with all

$$p(t), \frac{1}{p(t)}, q(t) \in L_\infty(a, x).$$

Let

$$\lambda_\beta > 0\ and\ \lambda_{\alpha_1}, \lambda_{\alpha_2} \ge 0,$$

such that

$$\lambda_\beta < p.$$

Set

$$P_i(s) := \int_0^s (s-t)^{\frac{p(\beta-\alpha_i-1)}{p-1}} \, (p(t+a))^{-1/(p-1)} \, dt, \ i = 1, 2; \ 0 \le s \le x-a, \tag{187}$$

$$A(s) := \frac{q(s+a)\,(P_1(s))^{\lambda_{\alpha_1}\left(\frac{p-1}{p}\right)} \ (P_2(s))^{\lambda_{\alpha_2}\left(\frac{p-1}{p}\right)} \, (p(s+a))^{-\lambda_\beta/p}}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}} \ (\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_2}}}, \tag{188}$$

$$A_0(x-a) := \left(\int_0^{x-a} (A(s))^{p/(p-\lambda_\beta)} \, ds \right)^{(p-\lambda_\beta)/p}, \tag{189}$$

and

$$\delta_1^* := \begin{cases} M^{1-\left((\lambda_{\alpha_1}+\lambda_\beta)/p\right)}, & if \ \ \lambda_{\alpha_1}+\lambda_\beta \le p, \\ \\ 2^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)-1}, & if \ \ \lambda_{\alpha_1}+\lambda_\beta \ge p. \end{cases} \tag{190}$$

Call

$$\varphi_1(x-a) := \left(A_0(x-a)\Big|_{\lambda_{\alpha_2}=0} \right) \left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta} \right)^{\lambda_\beta/p} \tag{191}$$

If $\lambda_{\alpha_2} = 0$, we obtain that,

$$\int_a^x q(s) \left(\sum_{j=1}^M |D_a^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}} \ |D_a^\beta f_j(s)|^{\lambda_\beta} \right) ds \le$$

$$\delta_1^* \varphi_1(x-a) \left[\int_a^x p(s) \left(\sum_{j=1}^M |D_a^\beta f_j(s)|^p \right) ds \right]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{192}$$

Proof
Similar to **Theorem 44**, transfer of **Theorem 4** of [3]. ■

Next we give

Theorem 80 All here as in **Theorem 79**. Denote

$$\delta_3 := \begin{cases} 2^{\lambda_{\alpha_2}/\lambda_\beta} - 1, & if \quad \lambda_{\alpha_2} \geq \lambda_\beta, \\ \\ 1, & if \quad \lambda_{\alpha_2} \leq \lambda_\beta. \end{cases} \tag{193}$$

$$\epsilon_2 := \begin{cases} 1, & if \quad \lambda_\beta + \lambda_{\alpha_2} \geq p, \\ \\ M^{1-\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)}, & if \quad \lambda_\beta + \lambda_{\alpha_2} \leq p, \end{cases} \tag{194}$$

and

$$\varphi_2(x-a) := \left(A_0(x-a)\Big|_{\lambda_{\alpha_1}=0} \right) 2^{\left(\frac{p-\lambda_\beta}{p}\right)} \left(\frac{\lambda_\beta}{\lambda_{\alpha_2} + \lambda_\beta} \right)^{\lambda_\beta/p} \delta_3^{(\lambda_\beta/p)}, \tag{195}$$

If $\lambda_{\alpha_1} = 0$, then it holds

$$\int_a^x q(s) \Big\{ \{ \sum_{j=1}^{M} [\, |D_a^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}} \; |D_a^\beta f_j(s)|^{\lambda_\beta} + |D_a^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}} \; |D_a^\beta f_{j+1}(s)|^{\lambda_\beta} \,] \} +$$

$$[\, |D_a^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}} \; |D_a^\beta f_1(s)|^{\lambda_\beta} + |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}} \; |D_a^\beta f_M(s)|^{\lambda_\beta} \,] \Big\} ds \leq$$

$$2^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)} \epsilon_2 \varphi_2(x-a) \Big\{ \int_a^x p(s) \Big(\sum_{j=1}^{M} |D_a^\beta f_j(s)|^p \Big) ds \Big\}^{\left(\frac{\lambda_\beta + \lambda_{\alpha_2}}{p}\right)} \tag{196}$$

Proof Usual transfer of **Theorem 5** of [3].
It follows the general case
Theorem 81 All here as in **Theorem 79**. Denote

$$\tilde{\gamma_1} := \begin{cases} 2^{((\lambda_{\alpha_1} + \lambda_{\alpha_2})/\lambda_\beta)} - 1, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} \geq \lambda_\beta, \\ \\ 1, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} \leq \lambda_\beta, \end{cases} \tag{197}$$

and

$$\tilde{\gamma_2} := \begin{cases} 1, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta \geq p, \\ \\ 2^{1-((\lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta)/p)}, & if \quad \lambda_{\alpha_1} + \lambda_{\alpha_2} + \lambda_\beta \leq p. \end{cases} \tag{198}$$

Set

$$\varphi_3(x-a) := A_0(x-a)\left(\frac{\lambda_\beta}{(\lambda_{\alpha_1}+\lambda_{\alpha_2})(\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)}\right)^{\lambda_\beta/p} \cdot [\lambda_{\alpha_1}^{\lambda_\beta/p}\tilde{\gamma_2}+2^{(p-\lambda_\beta)/p}\,(\tilde{\gamma_1}\lambda_{\alpha_2})^{\lambda_\beta/p}], \tag{199}$$

and

$$\epsilon_3 := \begin{cases} 1, \quad if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta \geq p, \\ M^{1-((\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta)/p)}, \quad if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta \leq p. \end{cases} \tag{200}$$

Then

$$\int_a^x q(s)\Big[\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}\ \ |D_a^{\beta}f_j(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}\ |D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_{j+1}(s)|^{\lambda_\beta}\ ]+$$

$$[\ |D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}\ \ |D_a^{\beta}f_1(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_M(s)|^{\lambda_\beta}\ ]\Big]ds \leq \tag{201}$$

$$2^{\left(\frac{\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta}{p}\right)}\epsilon_3\varphi_3(x-a)\Big\{\int_a^x p(s)\Big(\sum_{j=1}^{M}|D_a^{\beta}f_j(s)|^p\Big)ds\Big\}^{\left(\frac{\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta}{p}\right)}.$$

Proof
Usual transfer of **Theorem 6** of [3]. ■

We continue

Theorem 82 Let

$$\beta > \alpha_1 + 1,\ \alpha_1 \in \Re_+$$

and let

$$f_j \in L_1(a,x),\ j=1,\ldots,M \in \mathcal{N},\ a,\ x \in \Re,\ a<x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a,x]$, and let

$$D_a^{\beta-k}f_j(a)=0,\ for\ k=1,\ldots,[\beta]+1;\ j=1,\ldots,M.$$

Consider also

$$p(t) > 0 \ and \ q(t) \geq 0,$$

with all

$$p(t), \frac{1}{p(t)}, q(t) \in L_\infty(a,x).$$

Let

$$\lambda_\alpha \geq 0, \ 0 < \lambda_{\alpha+1} < 1, \ and \ p > 1.$$

Denote

$$\theta_3 := \begin{cases} 2^{\lambda_\alpha/(\lambda_{\alpha+1})} - 1, & if \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ \\ 1, & if \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases}$$

$$L(x-a) := \left(2 \int_a^x (q(s))^{(1/(1-\lambda_{\alpha+1}))} ds \right)^{(1-\lambda_{\alpha+1})} \left(\frac{\theta_3 \ \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}} \right)^{\lambda_{\alpha+1}}, \quad (202)$$

and

$$P_1(x-a) := \int_a^x (x-s)^{(\beta-\alpha_1-1)p/(p-1)} \ (p(s))^{-1/(p-1)} ds, \quad (203)$$

$$T(x-a) := L(x-a) \left(\frac{(P_1(x-a))^{\left(\frac{p-1}{p}\right)}}{\Gamma(\beta-\alpha_1)} \right)^{(\lambda_\alpha+\lambda_{\alpha+1})}, \quad (204)$$

and

$$w_1 := 2^{\left(\frac{p-1}{p}\right)(\lambda_\alpha+\lambda_{\alpha+1})}, \quad (205)$$

$$\Phi(x-a) := T(x-a) w_1. \quad (206)$$

Also put

$$\epsilon_4 := \begin{cases} 1, & if \quad \lambda_\alpha + \lambda_{\alpha+1} \geq p, \\ \\ M^{1-\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}, & if \quad \lambda_\alpha + \lambda_{\alpha+1} \leq p. \end{cases} \quad (207)$$

Then

$$\int_a^x q(s)\Big\{\{\sum_{j=1}^{M-1}[\,|D_a^{\alpha_1}f_j(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1}f_{j+1}(s)|^{\lambda_{\alpha+1}}+|D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1}f_j(s)|^{\lambda_{\alpha+1}}]\}+$$

$$[\ |D_a^{\alpha_1}f_1(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1}f_M(s)|^{\lambda_{\alpha+1}}+|D_a^{\alpha_1}f_M(s)|^{\lambda_\alpha}\ |D_a^{\alpha_1+1}f_1(s)|^{\lambda_{\alpha+1}}\ ]\Big\}ds\le$$

$$2^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)}\epsilon_4\phi(x-a)\Big[\int_a^x p(s)\Big(\sum_{j=1}^{M}|D_a^{\beta}f_j(s)|^p\Big)ds\Big]^{\left(\frac{\lambda_\alpha+\lambda_{\alpha+1}}{p}\right)} \tag{208}$$

Proof
Transfer of **Theorem 7** of [3]. ■

Next it comes

Theorem 83 All as in **Theorem 79**. Consider the special case of $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$. Denote

$$\tilde{T}(x-a):=A_0(x-a)\left(\frac{\lambda_\beta}{\lambda_{\alpha_1}+\lambda_\beta}\right)^{\lambda_\beta/p}\ 2^{(p-2\lambda_{\alpha_1}-3\lambda_\beta)/p}. \tag{209}$$

$$\epsilon_5:=\begin{cases}1, & if\ \ 2(\lambda_{\alpha_1}+\lambda_\beta)\ge p,\\ \\ M^{1-\left(\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}\right)}, & if\ \ 2(\lambda_{\alpha_1}+\lambda_\beta)\le p.\end{cases} \tag{210}$$

Then

$$\int_a^x q(s)\Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta}f_j(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_{j+1}(s)|^{\lambda_\beta}\ ]\}+$$

$$[\ |D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta}f_1(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_M(s)|^{\lambda_\beta}\ ]\Big\}ds\le$$

$$2^{\left(\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}\right)}\epsilon_5\tilde{T}(x-a)\Big[\int_a^x p(s)\Big(\sum_{j=1}^{M}|D_a^{\beta}f_j(s)|^p\Big)ds\Big]^{\left(\frac{2(\lambda_{\alpha_1}+\lambda_\beta)}{p}\right)}. \tag{211}$$

Proof
Transfer of **Theorem 8** of [3]. ■

Special cases follow

Corollary 84 (to **Theorem 79**, $\lambda_{\alpha_2} = 0,\ p(t) = q(t) = 1$.) It holds

$$\int_a^x \Big(\sum_{j=1}^{M} |D_a^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\beta} f_j(s)|^{\lambda_\beta} \Big) ds \le \delta_1^* \varphi_1(x-a) \Big[\int_a^x \sum_{j=1}^{M} [\ |D_a^{\beta} f_j(s)|^p\ ] ds \Big]^{\left(\frac{\lambda_{\alpha_1}+\lambda_\beta}{p}\right)}. \tag{212}$$

In (212) $\left(A_0(x-a)\Big|_{\lambda_{\alpha_2}=0} \right)$ of $\varphi_1(x)$ is given in **Corollary 49**, equation (123).

Proof

Transfer of **Corollary 9** of [3]. ■

Corollary 85 (to **Theorem 79**, $\lambda_{\alpha_2} = 0,\ p(t) = q(t) = 1, \lambda_{\alpha_1} = \lambda_\beta = 1,\ p = 2$.) In detail, let

$$\beta > \alpha_1,\ \alpha_1 \in \Re_+,\ \beta - \alpha_1 > (1/2),$$

and let

$$f_j \in L_1(a,x),\ j = 1, \ldots, M \in \mathcal{N},\ a,\ x \in \Re,\ a < x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a,x]$, and let

$$D_a^{\beta-k} f_j(a) = 0,\ for\ k = 1, \ldots, [\beta]+1;\ j = 1, \ldots, M.$$

Then

$$\int_a^x \Big(\sum_{j=1}^{M} |D_a^{\alpha_1} f_j(s)|\ |D_a^{\beta} f_j(s)| \Big) ds \le$$

$$\frac{(x-a)^{(\beta-\alpha_1)}}{2\Gamma(\beta-\alpha_1)\ \sqrt{\beta-\alpha_1}\ \sqrt{2\beta-2\alpha_1-1}} \Big\{ \int_a^x \Big[\sum_{j=1}^{M} (D_a^\beta f_j(s))^2 \Big] ds \Big\}. \tag{213}$$

Proof

Transfer of **Corollary 10** of [3]. ■

Corollary 86 (to **Theorem 80**, $\lambda_{\alpha_1} = 0,\ p(t) = q(t) = 1$.) It holds

$$\int_a^x \Big\{ \{ \sum_{j=1}^{M-1} [\ |D_a^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_j(s)|^{\lambda_\beta} + |D_a^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_{j+1}(s)|^{\lambda_\beta}] \} +$$

$$[\ |D_a^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_1(s)|^{\lambda_\beta} + |D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta} f_M(s)|^{\lambda_\beta}\ ] \Big\} ds \le$$

$$2^{\left(\frac{\lambda_\beta+\lambda_{\alpha_2}}{p}\right)}\epsilon_2\varphi_2(x-a)\Big\{\int_a^x\Big(\sum_{j=1}^M|D_a^\beta f_j(s)|^p\Big)ds\Big\}^{\left(\frac{\lambda_\beta+\lambda_{\alpha_2}}{p}\right)} \tag{214}$$

In (214), $\left(A_0(x-a)\Big|_{\lambda_{\alpha_1}=0}\right)$ of $\varphi_2(x-a)$ is given in **Corollary 51**, see equation (128).

Proof

Transfer of **Corollary 11** of [3]. $\blacksquare$

Corollary 87 (to **Theorem 80**, $\lambda_{\alpha_1}=0,\ p(t)=q(t)=1,\lambda_{\alpha_2}=\lambda_\beta=1,\ p=2$.) In detail, let

$$\alpha_2\in\Re_+,\ \beta>\alpha_2,\ \beta-\alpha_2>(1/2),$$

and let

$$f_j\in L_1(a,x),\ j=1,\ldots,M\in\mathcal{N},\ a,\ x\in\Re,\ a<x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a,x]$, and let

$$D_a^{\beta-k}f_j(a)=0,\ for\ k=1,\ldots,[\beta]+1;\ j=1,\ldots,M.$$

Then

$$\int_a^x\Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_2}f_{j+1}(s)|\ |D_a^\beta f_j(s)|+|D_a^{\alpha_2}f_j(s)|\ |D_a^\beta f_{j+1}(s)|]\}+$$

$$[\ |D_a^{\alpha_2}f_M(s)|\ |D_a^\beta f_1(s)|+|D_a^{\alpha_2}f_1(s)|\ |D_a^\beta f_M(s)|\ ]\Big\}ds$$

$$\leq\frac{\sqrt{2}(x-a)^{(\beta-\alpha_2)}}{\Gamma(\beta-\alpha_2)\ \sqrt{\beta-\alpha_2}\ \sqrt{2\beta-2\alpha_2-1}}\Big\{\int_a^x\Big[\sum_{j=1}^M(D_a^\beta f_j(s))^2\Big]ds\Big\}. \tag{215}$$

Proof

Transfer of **Corollary 12** of [3]. $\blacksquare$

Corollary 88 (to **Theorem 81**, $\lambda_{\alpha_1}=\lambda_{\alpha_2}=\lambda_\beta=1,\ p(t)=q(t)=1,\ p=3$.) It holds

$$\int_a^x\Big[\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1}f_j(s)|\ |D_a^{\alpha_2}f_{j+1}(s)|\ \ |D_a^\beta f_j(s)|+$$

$$|D_a^{\alpha_2}f_j(s)|\ |D_a^{\alpha_1}f_{j+1}(s)|\ \ |D_a^\beta f_{j+1}(s)|\ ]+[\ |D_a^{\alpha_1}f_1(s)|\ |D_a^{\alpha_2}f_M(s)|\ |D_a^\beta f_1(s)|+$$

$$|D_a^{\alpha_2} f_1(s)|\,|D_a^{\alpha_1} f_M(s)|\ \ |D_a^{\beta} f_M(s)|\,]\Big] ds \leq 2\varphi_3^*(x-a)\Big[\int_a^x \Big(\sum_{j=1}^{M} |D_a^{\beta} f_j(s)|^3\Big) ds\Big]. \tag{216}$$

Here

$$\varphi_3^*(x-a) := \left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{6}}\right) A_0(x-a), \tag{217}$$

where in this special case

$$A_0(x-a) = \frac{4(x-a)^{(2\beta-\alpha_1-\alpha_2)}}{\Gamma(\beta-\alpha_1)\ \Gamma(\beta-\alpha_2)\ [3(3\beta-3\alpha_1-1)(3\beta-3\alpha_2-1)(2\beta-\alpha_1-\alpha_2)\ ]^{2/3}}. \tag{218}$$

Proof
Transfer of **Corollary 13** of [3]. ■

Corollary 89 (to **Theorem 82**, $\lambda_\alpha = 1,\ \lambda_{\alpha+1} = 1/2,\ p(t) = q(t) = 1,\ p = 3/2.$) In detail, let

$$\alpha_1 \in \Re_+,\ \beta > \alpha_1 + 1,$$

and let

$$f_j \in L_1(a,x),\ j = 1,\ldots,M \in \mathcal{N},\ a,\ x \in \Re,\ a < x,$$

have, respectively, L_∞ fractional derivatives $D_a^{\beta} f_j$ in $[a,x]$, and let

$$D_a^{\beta-k} f_j(a) = 0,\ for\ k = 1,\ldots,[\beta]+1;\ j = 1,\ldots,M.$$

Set

$$\Phi^*(x-a) := \left(\frac{2}{\sqrt{3\beta-3\alpha_1-2}}\right) \cdot \frac{(x-a)^{\left(\frac{3\beta-3\alpha_1-1}{2}\right)}}{(\Gamma(\beta-\alpha_1))^{3/2}}. \tag{219}$$

Then

$$\int_a^x \Big\{\{\sum_{j=1}^{M-1} [\ |D_a^{\alpha_1} f_j(s)|\ \sqrt{|D_a^{\alpha_1+1} f_{j+1}(s)|} + |D_a^{\alpha_1} f_{j+1}(s)|\ \sqrt{|D_a^{\alpha_1+1} f_j(s)|}\ ]\} +$$

$$[\ |D_a^{\alpha_1} f_1(s)|\ \sqrt{|D_a^{\alpha_1+1} f_M(s)|} + |D_a^{\alpha_1} f_M(s)|\ \sqrt{|D_a^{\alpha_1+1} f_1(s)|}\ ]\Big\} ds \leq$$

$$2\Phi^*(x-a)\Big[\int_a^x \Big(\sum_{j=1}^{M} |D_a^{\beta} f_j(s)|^{3/2}\Big) ds\Big]. \tag{220}$$

Proof
Transfer of **Corollary 14** of [3]. ∎

Corollary 90 (to **Theorem 83**, $p = 2(\lambda_{\alpha_1} + \lambda_\beta) > 1,\ p(t) = q(t) = 1.$)
It holds

$$\int_a^x \Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta} f_j(s)|^{\lambda_\beta} +$$

$$|D_a^{\alpha_2} f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1} f_{j+1}(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta} f_{j+1}(s)|^{\lambda_\beta}\ ]\}+$$

$$[\ |D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta} f_1(s)|^{\lambda_\beta} +$$

$$|D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1} f_M(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta} f_M(s)|^{\lambda_\beta}\ ]\Big\} ds \le$$

$$2\tilde{T}(x-a)\Big[\int_a^x \Big(\sum_{j=1}^{M} |D_a^{\beta} f_j(s)|^{2(\lambda_{\alpha_1}+\lambda_\beta)}\Big) ds\Big]. \tag{221}$$

Here $\tilde{\tilde{T}}(x-a)$ in (221) is given by (209) and in detail by $\tilde{\tilde{T}}(x-a)$ of **Corollary 55** and equations (136)- (139).

Proof
Transfer of **Corollary 15** of [3]. ∎

Corollary 91 (to **Theorem 83**, $p = 4,\ \lambda_{\alpha_1} = \lambda_\beta = 1,\ p(t) = q(t) = 1.$)
It holds

$$\int_a^x \Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1} f_j(s)|\ (D_a^{\alpha_2} f_{j+1}(s))^2\ |D_a^{\beta} f_j(s)| +$$

$$(D_a^{\alpha_2} f_j(s))^2 |D_a^{\alpha_1} f_{j+1}(s)|\ \ |D_a^{\beta} f_{j+1}(s)|\ ]\}+[\ |D_a^{\alpha_1} f_1(s)|\ (D_a^{\alpha_2} f_M(s))^2\ |D_a^{\beta} f_1(s)| +$$

$$(D_a^{\alpha_2} f_1(s))^2\ |D_a^{\alpha_1} f_M(s)|\ \ |D_a^{\beta} f_M(s)|\ ]\Big\} ds \le 2\tilde{T}(x-a)\Big[\int_a^x \Big(\sum_{j=1}^{M} (D_a^{\beta} f_j(s))^4\Big) ds\Big]. \tag{222}$$

Here in (222) we have that $\tilde{T}(x-a) = T^*(x-a)$ of **Corollary 56**, see the equations (141)- (145).

Proof
Transfer of **Corollary 16** of [3]. ∎

Next we present the L_∞ case.

Theorem 92 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2$$

and let

$$f_j \in L_1(a,x),\quad j = 1, \ldots, M \in \mathcal{N},\ a,\ x \in \Re,\ a < x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a,x]$, and let

$$D_a^{\beta-k} f_j(a) = 0,\ for\ k = 1, \ldots, [\beta]+1;\ j = 1, \ldots,\ M.$$

Consider also

$$p(s) \geq 0,\ p(s) \in L_\infty(a,x).$$

Let $\lambda_{\alpha_1}, \lambda_{\alpha_2}, \lambda_\beta \geq 0$.
Set

$$\rho(x-a) = \frac{\|p(s)\|_\infty\ (x-a)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}[\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1]}. \tag{223}$$

Then

$$\int_a^x p(s)\Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1} f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_{j+1}(s)|^{\lambda_{\alpha_2}}\ \ |D_a^\beta f_j(s)|^{\lambda_\beta} +$$

$$|D_a^{\alpha_2} f_j(s)|^{\lambda_{\alpha_2}}\ |D_a^{\alpha_1} f_{j+1}(s)|^{\lambda_{\alpha_1}}\ \ |D_a^\beta f_{j+1}(s)|^{\lambda_\beta}\ ]\} +$$

$$[\ |D_a^{\alpha_1} f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2} f_M(s)|^{\lambda_{\alpha_2}}\ \ |D_a^\beta f_1(s)|^{\lambda_\beta} +$$

$$|D_a^{\alpha_2} f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\alpha_1} f_M(s)|^{\lambda_{\alpha_1}}\ \ |D_a^\beta f_M(s)|^{\lambda_\beta}\ ]\Big\}ds$$

$$\leq \rho(x-a)\Big\{\sum_{j=1}^{M}\{\|D_a^\beta f_j\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)} + \|D_a^\beta f_j\|_\infty^{2\lambda_{\alpha_2}}\}\Big\}. \tag{224}$$

Proof
Transfer of **Corollary 17** of [3]. ■

Similarly we give

Theorem 93 (as in **Theorem 92**; $\lambda_{\alpha_2} = 0$.) It holds

$$\int_a^x p(s)\Big(\sum_{j=1}^{M}|D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\beta}f_j(s)|^{\lambda_{\beta}}\Big)ds \leq$$

$$\frac{\|p(s)\|_\infty\ (x-a)^{(\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}[\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1]}\cdot\Big(\sum_{j=1}^{M}\|D_a^{\beta}f_j\|_\infty^{\lambda_{\alpha_1}+\lambda_\beta}\Big). \tag{225}$$

Proof
Based on **Theorem 18** of [3]. ■

It follows

Theorem 94 (as in **Theorem 92**; $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$.) It holds

$$\int_a^x p(s)\Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta}f_j(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_{j+1}(s)|^{\lambda_\beta}\ ]\}+$$

$$[\ |D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ \ |D_a^{\beta}f_1(s)|^{\lambda_\beta}+$$

$$|D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}+\lambda_\beta}\ |D_a^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\ \ |D_a^{\beta}f_M(s)|^{\lambda_\beta}\ ]\Big\}ds \leq$$

$$\leq\Big\{\frac{\|p(s)\|_\infty}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{(\lambda_{\alpha_1}+\lambda_\beta)}}$$

$$\cdot\frac{2(x-a)^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1)}}{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1)}\Big\}\cdot\Big(\sum_{j=1}^{M}\|D_a^{\beta}f_j\|_\infty^{2(\lambda_{\alpha_1}+\lambda_\beta)}\Big). \tag{226}$$

Proof
By **Theorem 19** of [3]. ■

We continue with

Theorem 95 (as in **Theorem 92**; $\lambda_\beta=0,\ \lambda_{\alpha_1}=\lambda_{\alpha_2}$.) It holds

$$\int_a^x p(s)\Big\{\{\sum_{j=1}^{M-1}[\ |D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_1}}+|D_a^{\alpha_2}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}]\}+$$

$$[\ |D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_2}f_M(s)|^{\lambda_{\alpha_1}}\ \ +|D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\ ]\Big\}ds$$

$$\leq 2\rho^*(x-a)\Big[\sum_{j=1}^{M}\|D_a^{\beta}f_j\|_\infty^{2\lambda_{\alpha_1}}\Big]. \tag{227}$$

Here we have

$$\rho^*(x-a) := \frac{(x-a)^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)}\ \|p(s)\|_\infty}{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}}}. \tag{228}$$

Proof
Based on **Theorem 20** of [3]. ∎

Next we give

Theorem 96 (as in **Theorem 92**, $\lambda_{\alpha_1}=0,\ \lambda_{\alpha_2}=\lambda_\beta$.) It holds

$$\int_a^x p(s)\Big\{\{\sum_{j=1}^{M-1}[\,|D_a^{\alpha_2}f_{j+1}(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta}f_j(s)|^{\lambda_{\alpha_2}}+|D_a^{\alpha_2}f_j(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta}f_{j+1}(s)|^{\lambda_{\alpha_2}}]\}+$$

$$[\,|D_a^{\alpha_2}f_M(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta}f_1(s)|^{\lambda_{\alpha_2}}+|D_a^{\alpha_2}f_1(s)|^{\lambda_{\alpha_2}}\ |D_a^{\beta}f_M(s)|^{\lambda_{\alpha_2}}\ ]\Big\}ds \le$$

$$2\left(\frac{(x-a)^{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)}\ \|p(s)\|_\infty}{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}}\right)\cdot\Big(\sum_{j=1}^{M}\|D_a^{\beta}f_j\|_\infty^{2\lambda_{\alpha_2}}\Big). \tag{229}$$

Proof
Based on **Theorem 21** of [3]. ∎

Some special cases follow.

Corollary 97 (to **Theorem 95**, all as in **Theorem 92**, $\lambda_\beta=0,\ \lambda_{\alpha_1}=\lambda_{\alpha_2},\ \alpha_2=\alpha_1+1$.) It holds

$$\int_a^x p(s)\Big\{\{\sum_{j=1}^{M-1}[\,|D_a^{\alpha_1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1+1}f_{j+1}(s)|^{\lambda_{\alpha_1}}+|D_a^{\alpha_1+1}f_j(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1}f_{j+1}(s)|^{\lambda_{\alpha_1}}]\}+$$

$$[\,|D_a^{\alpha_1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1+1}f_M(s)|^{\lambda_{\alpha_1}}\ +|D_a^{\alpha_1+1}f_1(s)|^{\lambda_{\alpha_1}}\ |D_a^{\alpha_1}f_M(s)|^{\lambda_{\alpha_1}}\ ]\Big\}ds \le$$

$$2\left(\frac{(x-a)^{(2\beta\lambda_{\alpha_1}-2\alpha_1\lambda_{\alpha_1}-\lambda_{\alpha_1}+1)}\ \|p(s)\|_\infty}{(2\beta\lambda_{\alpha_1}-2\alpha_1\lambda_{\alpha_1}-\lambda_{\alpha_1}+1)(\beta-\alpha_1)^{\lambda_{\alpha_1}}(\Gamma(\beta-\alpha_1))^{2\lambda_{\alpha_1}}}\right)\cdot\Big[\sum_{j=1}^{M}\|D_a^{\beta}f_j\|_\infty^{2\lambda_{\alpha_1}}\Big]. \tag{230}$$

Proof
Based on **Theorem 22** of [3]. ∎

Corollary 98 (to **Corollary 97**.) In detail, let

$$\alpha_1\in\Re_+,\ \beta>\alpha_1+1$$

and let

$$f_j \in L_1(a,x),\ j=1,\dots,M \in \mathcal{N},\ a,\ x \in \Re,\ a<x,$$

have, respectively, L_∞ fractional derivatives $D_a^\beta f_j$ in $[a,x]$, and let

$$D_a^{\beta-k} f_j(a) = 0,\ for\ k=1,\dots,[\beta]+1;\ j=1,\dots,M.$$

Then

$$\int_a^x \Big\{ \{ \sum_{j=1}^{M-1} [\ |D_a^{\alpha_1} f_j(s)|\ |D_a^{\alpha_1+1} f_{j+1}(s)| + |D_a^{\alpha_1} f_{j+1}(s)|\ |D_a^{\alpha_1+1} f_j(s)|\ ]\} +$$

$$[\ |D_a^{\alpha_1} f_1(s)|\ |D_a^{\alpha_1+1} f_M(s)| + |D_a^{\alpha_1} f_M(s)|\ |D_a^{\alpha_1+1} f_1(s)|\ ]\Big\} ds \le$$

$$\frac{(x-a)^{2(\beta-\alpha_1)}}{(\beta-\alpha_1)^2\ (\Gamma(\beta-\alpha_1))^2} \Big(\sum_{j=1}^M \|D_a^\beta f_j\|_\infty^2 \Big). \tag{231}$$

Proof
Based on **Corollary 23** of [3]. ∎

Corollary 99 (to **Corollary 98**.) It holds

$$\int_a^x \left(\sum_{j=1}^M |D_a^{\alpha_1} f_j(s)|\ |D_a^{\alpha_1+1} f_j(s)| \right) ds \le \frac{(x-a)^{2(\beta-\alpha_1)}}{2(\beta-\alpha_1)^2\ (\Gamma(\beta-\alpha_1))^2} \Big(\sum_{j=1}^M \|D_a^\beta f_j\|_\infty^2 \Big). \tag{232}$$

Proof
Based on inequality (155) of **Proposition 64**. ∎

Next we apply previous results of this subsection on the spherical shell A.

We make

Assumption 100 Let

$$\alpha_1, \alpha_2 \in \Re_+,\ \beta > \alpha_1, \alpha_2,\ \beta - \alpha_i > (1/p), p>1, i=1,2,$$

and for $j=1,\dots,\ M,\ M \in \mathcal{N}$, let $f_j \in L_1(A)$ with

$$\frac{\partial_{R_1}^\beta f_j(x)}{\partial r^\beta},\ \in L_\infty(A),\ x \in A,$$

$$A := B(0,R_2) - \overline{B(0,R_1)} \subseteq \Re^N,\ N \ge 2,\ 0<R_1<R_2.$$

Further assume that each $D_{R_1}^{\beta} f_j(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D_{R_1}^{\beta} f_j(rw)| \leq M_j$$

for some $M_j > 0$; $j = 1, \ldots, M$. For each $w \in S^{N-1} - (\cup_{j=1}^{M} K(f_j))$, we assume that $f_j(\cdot\, w)$ has an L_∞ fractional derivative $D_{R_1}^{\beta} f_j(\cdot\, w)$ in $[R_1, R_2]$, and that

$$D_{R_1}^{\beta-k} f_j(R_1 w) = 0,\ k = 1, \ldots, [\beta] + 1;$$

$j = 1, \ldots, M$. Let $\lambda_\beta > 0$ and $\lambda_{\alpha_1},\ \lambda_{\alpha_2} \geq 0$, such that $\lambda_\beta < p$. If $\alpha_1 = 0$ we set $\lambda_{\alpha_1} = 1$, and if $\alpha_2 = 0$ we set $\lambda_{\alpha_2} = 1$.

We need

Notation 101 (on **Assumption 100**.) We set

$$P_i(s) := \int_0^s (s-r)^{\frac{p(\beta-\alpha_i-1)}{p-1}} (r+R_1)^{\left(\frac{1-N}{p-1}\right)} dr,\ i = 1, 2;\ 0 \leq s \leq R_2 - R_1, \tag{233}$$

$$A(s) := \frac{(s+R_1)^{(N-1)\left(1-\left(\frac{\lambda_\beta}{p}\right)\right)} (P_1(s))^{\lambda_{\alpha_1}\left(\frac{p-1}{p}\right)} (P_2(s))^{\lambda_{\alpha_2}\left(\frac{p-1}{p}\right)}}{(\Gamma(\beta-\alpha_1))^{\lambda_{\alpha_1}} (\Gamma(\beta-\alpha_2))^{\lambda_{\alpha_2}}}, \tag{234}$$

and

$$A_0(R_2 - R_1) := \left(\int_0^{R_2-R_1} (A(s))^{p/(p-\lambda_\beta)} ds\right)^{(p-\lambda_\beta)/p}, \tag{235}$$

We present

Theorem 102 (All as in **Assumption 100** and **Notation 101**). Denote

$$\varphi_1(R_2 - R_1) := \left(A_0(R_2 - R_1)\Big|_{\lambda_{\alpha_2}=0}\right) \left(\frac{\lambda_\beta}{p}\right)^{\left(\frac{\lambda_\beta}{p}\right)} \tag{236}$$

Let $\lambda_{\alpha_1} > 0,\ \lambda_{\alpha_2} = 0$ and $p = \lambda_{\alpha_1} + \lambda_\beta > 1$. Then

$$\int_A \Big[\sum_{j=1}^{M} \Big|\frac{\partial_{R_1}^{\alpha_1} f_j(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}} \Big|\frac{\partial_{R_1}^{\beta} f_j(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big] dx \leq$$

$$\varphi_1(R_2-R_1)\Big[\int_A\Big(\sum_{j=1}^{M}\Big|\frac{\partial^{\beta}_{R_1}f_j(x)}{\partial r^{\beta}}\Big|^{p}\Big)dx\Big]. \tag{237}$$

Proof
Based on **Theorem 79** and similar to the proof of **Theorem 67**, notice here $\lambda_{S^{N-1}}(\cup_{j=1}^{M}K(f_j))=0$. ■

Next we give

Theorem 103 (All as in **Assumption 100** and **Notation 101**). We denote

$$\delta_3:=\begin{cases}2^{\lambda_{\alpha_2}/\lambda_\beta}-1, & if \quad \lambda_{\alpha_2}\geq\lambda_\beta,\\ \\ 1, & if \quad \lambda_{\alpha_2}\leq\lambda_\beta,\end{cases} \tag{238}$$

and

$$\varphi_2(R_2-R_1):=\Big(A_0(R_2-R_1)\Big|_{\lambda_{\alpha_1}=0}\Big)2^{\left(\frac{\lambda_{\alpha_2}}{p}\right)}\left(\frac{\lambda_\beta}{p}\right)^{(\lambda_\beta/p)}\delta_3^{(\lambda_\beta/p)}, \tag{239}$$

Here $\lambda_{\alpha_1}=0,\ \lambda_{\alpha_2}>0$ and $p=\lambda_\beta+\lambda_{\alpha_2}>1$. Then

$$\int_A\Big\{\{\sum_{j=1}^{M-1}\Big[\Big|\frac{\partial^{\alpha_2}_{R_1}f_{j+1}(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\Big|\frac{\partial^{\beta}_{R_1}f_j(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+\Big|\frac{\partial^{\alpha_2}_{R_1}f_j(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\Big|\frac{\partial^{\beta}_{R_1}f_{j+1}(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\}+$$

$$\Big[\Big|\frac{\partial^{\alpha_2}_{R_1}f_M(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\Big|\frac{\partial^{\beta}_{R_1}f_1(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+\Big|\frac{\partial^{\alpha_2}_{R_1}f_1(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\Big|\frac{\partial^{\beta}_{R_1}f_M(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]dx\Big\}\leq$$

$$2\varphi_2(R_2-R_1)\Big[\int_A\Big(\sum_{j=1}^{M}\Big|\frac{\partial^{\beta}_{R_1}f_j(x)}{\partial r^{\beta}}\Big|^{p}\Big)dx\Big]. \tag{240}$$

Proof
Based on **Theorem 80** and similar to the proof of **Theorem 67**. ■

It follows the general case

Theorem 104 All as in **Assumption 100** and **Notation 101**. Here $\lambda_{\alpha_1},\ \lambda_{\alpha_2}>0$, $p=\lambda_{\alpha_1}+\lambda_{\alpha_2}+\lambda_\beta>1$. Denote

$$\tilde{\gamma_1}:=\begin{cases}2^{((\lambda_{\alpha_1}+\lambda_{\alpha_2})/\lambda_\beta)}-1, & if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}\geq\lambda_\beta,\\ \\ 1, & if \quad \lambda_{\alpha_1}+\lambda_{\alpha_2}\leq\lambda_\beta,\end{cases} \tag{241}$$

and

$$\varphi_3(R_2-R_1) := A_0(R_2-R_1)\left(\frac{\lambda_\beta}{(\lambda_{\alpha_1}+\lambda_{\alpha_2})p}\right)^{(\lambda_\beta/p)}[\lambda_{\alpha_1}{}^{(\lambda_\beta/p)} + 2^{(\lambda_{\alpha_1}+\lambda_{\alpha_2})/p}(\tilde{\gamma}_1\lambda_{\alpha_2})^{(\lambda_\beta/p)}]. \tag{242}$$

Then

$$\int_A \Big\{ \sum_{j=1}^{M-1} \Big[\Big| \frac{\partial^{\alpha_1}_{R_1} f_j(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial^{\alpha_2}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial^{\beta}_{R_1} f_j(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial^{\alpha_2}_{R_1} f_j(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial^{\alpha_1}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial^{\beta}_{R_1} f_{j+1}(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] +$$

$$\Big[\Big| \frac{\partial^{\alpha_1}_{R_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial^{\alpha_2}_{R_1} f_M(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial^{\alpha_2}_{R_1} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial^{\alpha_1}_{R_1} f_M(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial^{\beta}_{R_1} f_M(x)}{\partial r^{\beta}} \Big|^{\lambda_\beta} \Big] \Big\} dx$$

$$\leq 2\varphi_3(R_2-R_1)\Big[\int_A \Big(\sum_{j=1}^{M} \Big| \frac{\partial^{\beta}_{R_1} f_j(x)}{\partial r^{\beta}} \Big|^{p} \Big) dx \Big]. \tag{243}$$

Proof
Based on **Theorem 81** and similar to the proof of **Theorem 67**. ■

We continue with

Theorem 105 All as in **Assumption 100**. Here $\alpha_2 = \alpha_1 + 1$, $\lambda_\alpha := \lambda_{\alpha_1} \geq 0$, $\lambda_{\alpha+1} := \lambda_{\alpha_2} \in (0,1)$ and $p = \lambda_\alpha + \lambda_{\alpha_1} > 1$. Denote

$$\theta_3 := \begin{cases} 2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1, & if \quad \lambda_\alpha \geq \lambda_{\alpha+1}, \\ \\ 1, & if \quad \lambda_\alpha \leq \lambda_{\alpha+1}, \end{cases} \tag{244}$$

$$L(R_2-R_1) := \left[2\,\frac{(1-\lambda_{\alpha+1})}{(N-\lambda_{\alpha+1})}\left(R_2^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}} - R_1^{\frac{N-\lambda_{\alpha+1}}{1-\lambda_{\alpha+1}}}\right)\right]^{(1-\lambda_{\alpha+1})}\left(\frac{\theta_3\ \lambda_{\alpha+1}}{p}\right)^{\lambda_{\alpha+1}}, \tag{245}$$

and

$$P_1(R_2-R_1) := \int_{R_1}^{R_2} (R_2-r)^{(\beta-\alpha_1-1)p/(p-1)}\ r^{(1-N)/(p-1)} dr, \tag{246}$$

and

$$\Phi(R_2-R_1) := L(R_2-R_1)\left(\frac{(P_1(R_2-R_1))^{(p-1)}}{(\Gamma(\beta-\alpha_1))^p}\right) 2^{(p-1)}. \tag{247}$$

Then

$$\int_A \Big\{\{\sum_{j=1}^{M-1}\Big[\Big|\frac{\partial_{R_1}^{\alpha_1} f_j(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_\alpha}\Big|\frac{\partial_{R_1}^{\alpha_1+1} f_{j+1}(x)}{\partial r^{\alpha_1+1}}\Big|^{\lambda_{\alpha+1}}+\Big|\frac{\partial_{R_1}^{\alpha_1} f_{j+1}(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_\alpha}\Big|\frac{\partial_{R_1}^{\alpha_1+1} f_j(x)}{\partial r^{\alpha_1+1}}\Big|^{\lambda_{\alpha+1}}\Big]\}$$

$$+\Big[\Big|\frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_\alpha}\Big|\frac{\partial_{R_1}^{\alpha_1+1} f_M(x)}{\partial r^{\alpha_1+1}}\Big|^{\lambda_{\alpha+1}}+\Big|\frac{\partial_{R_1}^{\alpha_1} f_M(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_\alpha}\Big|\frac{\partial_{R_1}^{\alpha_1+1} f_1(x)}{\partial r^{\alpha_1+1}}\Big|^{\lambda_{\alpha+1}}\Big]\Big\}dx$$

$$\leq 2\Phi(R_2-R_1)\Big[\int_A\Big(\sum_{j=1}^{M}\Big|\frac{\partial_{R_1}^{\beta} f_j(x)}{\partial r^{\beta}}\Big|^p\Big)dx\Big]. \tag{248}$$

Proof
Based on **Theorem 82**, similar to the proof **Theorem 67**. ■

We also give

Theorem 106 (All as in **Assumption 100** and **Notation 101**). Here $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$ and $p=2(\lambda_{\alpha_1}+\lambda_\beta)>1$. Denote

$$\tilde{T}(R_2-R_1) := A_0(R_2-R_1)\left(\frac{2\lambda_\beta}{p}\right)^{\lambda_\beta/p} 2^{-\lambda_\beta/p}. \tag{249}$$

Then

$$\int_A \Big\{\{\sum_{j=1}^{M-1}\Big[\Big|\frac{\partial_{R_1}^{\alpha_1} f_j(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\Big|\frac{\partial_{R_1}^{\alpha_2} f_{j+1}(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\Big|\frac{\partial_{R_1}^{\beta} f_j(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+$$

$$\Big|\frac{\partial_{R_1}^{\alpha_2} f_j(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\Big|\frac{\partial_{R_1}^{\alpha_1} f_{j+1}(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\Big|\frac{\partial_{R_1}^{\beta} f_{j+1}(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\}+$$

$$\Big[\Big|\frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\Big|\frac{\partial_{R_1}^{\alpha_2} f_M(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\Big|\frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+$$

$$\Big|\frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\Big|\frac{\partial_{R_1}^{\alpha_1} f_M(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\Big|\frac{\partial_{R_1}^{\beta} f_M(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\Big\}dx$$

$$\leq 2\tilde{T}(R_2-R_1)\left[\int_A\Big(\sum_{j=1}^{M}\Big|\frac{\partial_{R_1}^{\beta} f_j(x)}{\partial r^{\beta}}\Big|^p\Big)dx\right]. \tag{250}$$

Proof
Based on **Theorem 83**. ■

Next we give L_∞ results on the shell A involving several functions. We make

Assumption 107 Let

$$\alpha_1, \alpha_2 \in \Re_+, \ \beta > \alpha_1, \alpha_2$$

and for $j = 1, \ldots, \ M, \ M \in \mathcal{N}$, let $f_j \in L_1(A)$ with

$$\frac{\partial^\beta_{R_1} f_j(x)}{\partial r^\beta}, \ \in L_\infty(A), \ x \in A,$$

$$A := B(0, R_2) - \overline{B(0, R_1)} \subseteq \Re^N, \ N \geq 2, \ 0 < R_1 < R_2.$$

Further assume that each $D^\beta_{R_1} f_j(rw) \in \Re$ for almost all $r \in [R_1, R_2]$, for each $w \in S^{N-1}$, and for these

$$|D^\beta_{R_1} f_j(rw)| \leq M_j$$

for some $M_j > 0$; $j = 1, \ldots, \ M$. For each $w \in S^{N-1} - (\cup_{j=1}^M K(f_j))$, we assume that $f_j(\cdot \ w)$ has an L_∞ fractional derivative $D^\beta_{R_1} f_j(\cdot \ w)$ in $[R_1, R_2]$, and that

$$D^{\beta-k}_{R_1} f_j(R_1 w) = 0, \ k = 1, \ldots, [\beta] + 1;$$

$j = 1, \ldots, \ M$. Let $\lambda_{\alpha_1}, \ \lambda_{\alpha_2}, \ \lambda_\beta \geq 0$. If $\alpha_1 = 0$ we set $\lambda_{\alpha_1} = 1$, and if $\alpha_2 = 0$ we set $\lambda_{\alpha_2} = 1$.

We present

Theorem 108 All here as in **Assumption 107**. Set

$$\rho(R_2 - R_1) = \frac{{R_2}^{N-1} (R_2 - R_1)^{(\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}}{(\Gamma(\beta - \alpha_1 + 1))^{\lambda_{\alpha_1}} (\Gamma(\beta - \alpha_2 + 1))^{\lambda_{\alpha_2}} (\beta\lambda_{\alpha_1} - \alpha_1\lambda_{\alpha_1} + \beta\lambda_{\alpha_2} - \alpha_2\lambda_{\alpha_2} + 1)}. \tag{251}$$

Then

$$\int_A \Big\{ \{ \sum_{j=1}^{M-1} \Big[\Big| \frac{\partial^{\alpha_1}_{R_1} f_j(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial^{\alpha_2}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \ \Big| \frac{\partial^\beta_{R_1} f_j(x)}{\partial r^\beta} \Big|^{\lambda_\beta} +$$

$$\Big| \frac{\partial^{\alpha_2}_{R_1} f_j(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \ \Big| \frac{\partial^{\alpha_1}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \ \Big| \frac{\partial^\beta_{R_1} f_{j+1}(x)}{\partial r^\beta} \Big|^{\lambda_\beta} \Big] \} +$$

$$\Big[\Big|\frac{\partial^{\alpha_1}_{R_1} f_1(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\alpha_2}_{R_1} f_M(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\ \Big|\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+$$

$$\Big|\frac{\partial^{\alpha_2}_{R_1} f_1(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_2}}\ \Big|\frac{\partial^{\alpha_1}_{R_1} f_M(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\beta}_{R_1} f_M(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\Big\}dx$$

$$\leq \frac{2\pi^{N/2}}{\Gamma(N/2)}\rho(R_2-R_1)\Big\{\sum_{j=1}^{M}[{M_j}^{2(\lambda_{\alpha_1}+\lambda_\beta)}+{M_j}^{2\lambda_{\alpha_2}}]\Big\}. \tag{252}$$

Proof
Based on **Theorem 92**, similar proof as in **Theorem 73**. ∎

Similarly we give

Theorem 109 All as in **Assumption 107**, $\lambda_{\alpha_2}=0$). Then

$$\int_A\Big[\sum_{j=1}^{M}\Big|\frac{\partial^{\alpha_1}_{R_1} f_j(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\beta}_{R_1} f_j(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]dx\leq$$

$$\frac{{R_2}^{N-1}(R_2-R_1)^{(\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}\ [\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+1]}\left(\sum_{j=1}^{M}{M_j}^{(\lambda_{\alpha_1}+\lambda_\beta)}\right)\frac{2\pi^{N/2}}{\Gamma(N/2)}. \tag{253}$$

Proof
Based on **Theorem 93**, similar to **Theorem 73**. ∎

It follows

Theorem 110 All as in **Assumption 107**, $\lambda_{\alpha_2}=\lambda_{\alpha_1}+\lambda_\beta$. Then

$$\int_A\Big\{\{\sum_{j=1}^{M-1}\Big[\Big|\frac{\partial^{\alpha_1}_{R_1} f_j(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\alpha_2}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\ \Big|\frac{\partial^{\beta}_{R_1} f_j(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+$$

$$\Big|\frac{\partial^{\alpha_2}_{R_1} f_j(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\ \Big|\frac{\partial^{\alpha_1}_{R_1} f_{j+1}(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\beta}_{R_1} f_{j+1}(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\}$$

$$+\Big[\Big|\frac{\partial^{\alpha_1}_{R_1} f_1(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\alpha_2}_{R_1} f_M(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\ \Big|\frac{\partial^{\beta}_{R_1} f_1(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}+$$

$$\Big|\frac{\partial^{\alpha_2}_{R_1} f_1(x)}{\partial r^{\alpha_2}}\Big|^{\lambda_{\alpha_1}+\lambda_\beta}\ \Big|\frac{\partial^{\alpha_1}_{R_1} f_M(x)}{\partial r^{\alpha_1}}\Big|^{\lambda_{\alpha_1}}\ \Big|\frac{\partial^{\beta}_{R_1} f_M(x)}{\partial r^{\beta}}\Big|^{\lambda_\beta}\Big]\Big\}dx\leq$$

$$\frac{{R_2}^{N-1}(R_2-R_1)^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1)}}{(\Gamma(\beta-\alpha_1+1))^{\lambda_{\alpha_1}}\ (\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}+\lambda_\beta}[2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}+\beta\lambda_\beta-\alpha_2\lambda_{\alpha_1}-\alpha_2\lambda_\beta+1]}.$$

$$\left[\sum_{j=1}^{M} {M_j}^{2(\lambda_{\alpha_1}+\lambda_\beta)}\right] \frac{4\pi^{N/2}}{\Gamma(N/2)}. \tag{254}$$

Proof
Based on **Theorem 94** and similar to the proof of **Theorem 73**. ■

We continue with

Theorem 111 All as in **Assumption 107**, here $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2}$. Then

$$\int_A \Big\{ \Big\{ \sum_{j=1}^{M-1} \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_j(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_{j+1}(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_j(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_{j+1}(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] \Big\} +$$

$$\Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_2} f_M(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_M(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] \Big\} dx \leq \frac{4\pi^{N/2}}{\Gamma(N/2)}$$

$$\left(\frac{{R_2}^{N-1}(R_2-R_1)^{(2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)}}{(\Gamma(\beta-\alpha_1+1)\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_1}}\ (2\beta\lambda_{\alpha_1}-\alpha_1\lambda_{\alpha_1}-\alpha_2\lambda_{\alpha_1}+1)} \right) \left(\sum_{j=1}^{M} {M_j}^{2\lambda_{\alpha_1}} \right). \tag{255}$$

Proof
Based on **Theorem 95**. ■

Next we give

Theorem 112 All as in **Assumption 107**, here $\lambda_{\alpha_1} = 0,\ \lambda_{\alpha_2} = \lambda_\beta$. Then

$$\int_A \Big\{ \Big\{ \sum_{j=1}^{M-1} \Big[\Big| \frac{\partial_{R_1}^{\alpha_2} f_{j+1}(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_j(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_j(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_{j+1}(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} \Big] \Big\} +$$

$$\Big[\Big| \frac{\partial_{R_1}^{\alpha_2} f_M(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_1(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} + \Big| \frac{\partial_{R_1}^{\alpha_2} f_1(x)}{\partial r^{\alpha_2}} \Big|^{\lambda_{\alpha_2}} \Big| \frac{\partial_{R_1}^{\beta} f_M(x)}{\partial r^{\beta}} \Big|^{\lambda_{\alpha_2}} \Big] \Big\} dx \leq \frac{4\pi^{N/2}}{\Gamma(N/2)}$$

$$\left(\frac{{R_2}^{N-1}(R_2-R_1)^{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)}}{(\beta\lambda_{\alpha_2}-\alpha_2\lambda_{\alpha_2}+1)(\Gamma(\beta-\alpha_2+1))^{\lambda_{\alpha_2}}} \right) \left(\sum_{j=1}^{M} {M_j}^{2\lambda_{\alpha_2}} \right). \tag{256}$$

Proof
Based on **Theorem 96**. ■

We finish this section with a special case.

Corollary 113 (to **Theorem 111**) All as in **Assumption 107**, here $\lambda_\beta = 0,\ \lambda_{\alpha_1} = \lambda_{\alpha_2},\ \alpha_2 = \alpha_1 + 1$. Then

$$\int_A \Big\{ \{ \sum_{j=1}^{M-1} \Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_j(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_{j+1}(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_j(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_{j+1}(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] \} +$$

$$\Big[\Big| \frac{\partial_{R_1}^{\alpha_1} f_1(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_M(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} + \Big| \frac{\partial_{R_1}^{\alpha_1+1} f_1(x)}{\partial r^{\alpha_1+1}} \Big|^{\lambda_{\alpha_1}} \Big| \frac{\partial_{R_1}^{\alpha_1} f_M(x)}{\partial r^{\alpha_1}} \Big|^{\lambda_{\alpha_1}} \Big] \Big\} dx \leq \frac{4\pi^{N/2}}{\Gamma(N/2)}$$

$$\left(\frac{R_2{}^{N-1}(R_2 - R_1)^{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)}}{(2\beta\lambda_{\alpha_1} - 2\alpha_1\lambda_{\alpha_1} - \lambda_{\alpha_1} + 1)(\beta - \alpha_1)^{\lambda_{\alpha_1}} (\Gamma(\beta - \alpha_1))^{2\lambda_{\alpha_1}}} \right) \left(\sum_{j=1}^{M} M_j{}^{2\lambda_{\alpha_1}} \right). \tag{257}$$

Proof
Based on **Corollary 97**. ■

We finish the article with the proof that $D^\alpha f$ of **Lemma 7**, see (8), also other similar fractional derivatives here, are such that

$$D^\alpha f \in AC([0,x]) \ for \ \beta - \alpha \geq 1$$

and

$$D^\alpha f \in C([0,x]), \ for \ \beta - \alpha \in (0,1).$$

The last derive from the next

Proposition 114 Let $r > 0$, $F \in L_\infty(a,b)$ and

$$G(s) := \int_a^s (s-t)^{r-1} F(t) dt, \tag{258}$$

all $s \in [a,b]$. Then

$$G \in AC([a,b]) \ for \ r \geq 1$$

and

$$G \in C([a,b]), \ only \ for \ r \in (0,1).$$

Proof
1)Case $r \geq 1$.

We use the definition of absolute continuity. So for every $\epsilon > 0$ we need $\delta > 0$: whenever (a_i, b_i), $i = 1, \ldots, n$, are disjoint open subintervals of $[a,b]$, then

$$\sum_{i=1}^{n}(b_i - a_i) < \delta \ \Rightarrow \ \sum_{i=1}^{n}|G(b_i) - G(a_i)| < \epsilon.$$

If $\|F\|_\infty = 0$, then $G(s) = 0$, for all $s \in \ [a, b]$, the trivial case and all fulfilled. So we assume $\|F\|_\infty \neq 0$.

Hence we have

$$G(b_i) - G(a_i) = \int_{a_i}^{b_i}(b_i - t)^{r-1}F(t)dt - \int_{a}^{a_i}(a_i - t)^{r-1}F(t)dt =$$

$$\int_{a}^{a_i}(b_i - t)^{r-1}F(t)dt - \int_{a}^{a_i}(a_i - t)^{r-1}F(t)dt + \int_{a_i}^{b_i}(b_i - t)^{r-1}F(t)dt =$$

$$\int_{a}^{a_i}\left((b_i - t)^{r-1} - (a_i - t)^{r-1}\right)F(t)dt + \int_{a_i}^{b_i}(b_i - t)^{r-1}F(t)dt. \tag{259}$$

Call

$$I_i := \int_{a}^{a_i}|(b_i - t)^{r-1} - (a_i - t)^{r-1}|dt. \tag{260}$$

Thus

$$|G(b_i) - G(a_i)| \leq \left[I_i + \frac{(b_i - a_i)^r}{r}\right]\|F\|_\infty := T_i. \tag{261}$$

If $r = 1$, then $I_i = 0$, and

$$|G(b_i) - G(a_i)| \leq \|F\|_\infty(b_i - a_i), \tag{262}$$

for all $i := 1, \ldots, n$.

If $r > 1$, then since $\left[(b_i - t)^{r-1} - (a_i - t)^{r-1}\right] \geq 0$, for all $t \in [a, a_i]$, we find

$$I_i = \int_{a}^{a_i}\left((b_i - t)^{r-1} - (a_i - t)^{r-1}\right)dt = \frac{(b_i - a)^r - (a_i - a)^r - (b_i - a_i)^r}{r}$$

$$= \frac{r(\xi - a)^{r-1}(b_i - a_i) - (b_i - a_i)^r}{r}, \ for \ some \ \xi \in (a_i, b_i). \tag{263}$$

Therefore, it holds

$$I_i \leq \frac{r(b - a)^{r-1}(b_i - a_i) - (b_i - a_i)^r}{r}, \tag{264}$$

and

$$\left(I_i + \frac{(b_i - a_i)^r}{r}\right) \leq (b-a)^{r-1}(b_i - a_i). \tag{265}$$

That is

$$T_i \leq \|F\|_\infty (b-a)^{r-1}(b_i - a_i),$$

so that

$$|G(b_i) - G(a_i)| \leq \|F\|_\infty (b-a)^{r-1}(b_i - a_i), \;\; for\ all\ i = 1, \ldots, n. \tag{266}$$

So in the case of $r = 1$, and by choosing $\delta := \frac{\epsilon}{\|F\|_\infty}$, we get

$$\sum_{i=1}^{n} |G(b_i) - G(a_i)| \leq^{(262)} \|F\|_\infty \left(\sum_{i=1}^{n} (b_i - a_i)\right) \leq \|F\|_\infty \delta = \epsilon, \tag{267}$$

proving for $r = 1$ that G is absolutely continuous. In the case of $r > 1$, and by choosing $\delta := \frac{\epsilon}{\|F\|_\infty (b-a)^{r-1}}$, we get

$$\sum_{i=1}^{n} |G(b_i) - G(a_i)| \leq^{(266)} \|F\|_\infty (b-a)^{r-1} \left(\sum_{i=1}^{n} (b_i - a_i)\right) \tag{268}$$

$$\leq \|F\|_\infty (b-a)^{r-1} \delta = \epsilon,$$

proving for $r > 1$ that G is absolutely continuous again.

2) Case of $0 < r < 1$. Let $a_{i_*}, b_{i_*} \in [a, b] : \; a_{i_*} \leq b_{i_*}$. Then $(a_{i_*} - t)^{r-1} \geq (b_{i_*} - t)^{r-1}$, for all $t \in [a, a_{i_*}]$. Then

$$I_{i_*} = \int_a^{a_{i_*}} \left((a_{i_*} - t)^{r-1} - (b_{i_*} - t)^{r-1}\right) dt = \frac{(b_{i_*} - a_{i_*})^r}{r} +$$

$$\left(\frac{(a_{i_*} - a)^r - (b_{i_*} - a)^r}{r}\right) \leq \frac{(b_{i_*} - a_{i_*})^r}{r}, \tag{269}$$

by $(a_{i_*} - a)^r - (b_{i_*} - a)^r < 0$.

I.e.

$$I_{i_*} \leq \frac{(b_{i_*} - a_{i_*})^r}{r} \tag{270}$$

and

$$T_{i_*} \leq \frac{2(b_{i_*} - a_{i_*})^r}{r} \, \|F\|_\infty, \tag{271}$$

proving that

$$|G(b_{i_*}) - G(a_{i_*})| \leq \left(\frac{2\|F\|_\infty}{r} \right) (b_{i_*} - a_{i_*})^r, \tag{272}$$

which is proving that G is continuous.

Taking the special case of $a = 0$ and $F(t) = 1$, for all $t \in [0, b]$, we get that

$$G(s) = \frac{s^r}{r}, \ all \ s \in [0, b], \ for \ 0 < r < 1. \tag{273}$$

The last is a Lipschitz function of order $r \in (0, 1)$, which is not absolutely continuous. Consequently G for $r \in (0, 1)$ in general, cannot be absolutely continuous. That completes the proof. ■

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